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**Very singular solutions of odd-order PDEs, with linear and nonlinear dispersion**

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# Very Singular Solutions of Odd-Order PDEs, with Linear and Nonlinear Dispersion

submitted by

Ray Stephen Fernandes

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

November 2008

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Ray Stephen Fernandes

# Summary

Asymptotic properties of solutions of the linear dispersion equation

$$u_t = u_{xxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

and its  $(2k+1)$ th-order generalisations are studied. General Hermitian spectral theory and asymptotic behaviour of its kernel, for the rescaled operator

$$\mathbf{B} = D^3 + \frac{1}{3} y D_y + \frac{1}{3} I,$$

is developed, where a complete set of bi-orthonormal pair of eigenfunctions,  $\{\psi_\beta\}, \{\psi_\beta^*\}$ , are found. The results apply to the construction of VSS (very singular solutions) of the semilinear equation with absorption

$$u_t = u_{xxx} - |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad \text{where} \quad p > 1,$$

which serves as a basic model for various applications, including the classic KdV area.

Finally, the nonlinear dispersion equations such as

$$u_t = (|u|^n u)_{xxx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

and

$$u_t = (|u|^n u)_{xxx} - |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,$$

where  $n > 0$ , are studied and their “nonlinear eigenfunctions” are constructed. The basic tools include numerical methods and “homotopy-deformation” approaches, where the limits  $n \rightarrow 0$  and  $n \rightarrow +\infty$  turn out to be fruitful. Local existence and uniqueness is proved and some bounds on the highly oscillatory tail are found.

These odd-order models were not treated in existing mathematical literature, from the proposed point of view.

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# Chapter 1

## Introduction

### 1.1 General Semilinear Odd-Order Models and Some History

As a first basic model, we will study higher odd-order partial differential equations (PDEs), of the form

$$u_t = (-1)^{[\frac{m}{2}+1]} D_x^m u + \tilde{g}(u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad (1.1)$$

with initial data  $u(x, 0) = u_0$ , and where  $m = 3, 5, 7, \dots$  is an odd integer. Here  $D_x^m$  denotes the  $m^{th}$  partial derivative with respect to the spatial variable  $x$  and  $u_t$  denotes the partial derivative of  $u(x, t)$  with respect to the time variable  $t$ . The brackets  $[\cdot]$ , in this equation, denote the integer part, so that (1.1) also includes semilinear parabolic equations for even  $m = 2, 4, 6, \dots$ . The analogy between odd and even-order PDEs, such as (1.1), is rather fruitful and will be used later on.

The function  $\tilde{g}(u)$  usually corresponds to some absorption-reaction type phenomena and sometimes is assumed to include differential terms, such as  $D_x^{\tilde{m}} \tilde{f}(u)$ , with  $\tilde{m} < m$  (although we do not consider such cases). Note that any constant coefficients may be placed in front of any of the terms, as these can easily be scaled out to obtain a PDE of the same form. It is worth mentioning that, besides special and completely integrable PDEs, general odd-order models such as (1.1) are much less studied in the mathematical literature, than the parabolic even-order ones.

As already stated, the focus is on odd-order PDEs in particular, with the



general higher-order model (1.1) for  $m = 2k + 1$ , so it takes the form

$$u_t = (-1)^{k+1} D_x^{2k+1} u + \tilde{g}(u).$$

This is a generalised PDE of order  $2k + 1$ , as this is the order of the highest derivative involved.

The most classical and well-known example of such an odd-order equation is the KdV equation, given by

$$u_t = u_{xxx} + uu_x. \tag{1.2}$$

The KdV equation models long waves in shallow water; see references below.

### 1.1.1 History of the KdV Equation

The history behind the KdV equation originates from one of the most well known and amazing stories in fluid mechanics. The importance of solitary waves (which led to the discovery of the KdV equation), was first brought forward by John Scott Russell in 1834, whilst observing a wave in a canal near Edinburgh. During the 19th century, the study of water waves was of particular importance, especially in the application of naval architecture and knowledge of floods and tides. Modelling of these waves using PDEs was important in the understanding of their structure and behaviour.

Russell had been following a barge towed by horses along the Union canal. The barge apparently stopped, but however a small body of water was set in motion and formed a wave, which carried on down the canal. The wave was noted to move at about 8 miles an hour and was a couple of feet in height and about 30 feet in length. Russell followed the wave, which appeared not to change shape nor speed. After losing track of the wave, in the windings of the channel, he decided to later conduct experiments to try to study this phenomena more carefully. Russell always thought that the solitary wave which he had seen, which he called the wave of translation, was of fundamental importance. However many scientists at that time disagreed. Whilst Russell himself could not prove existence of such waves, he is still well known for his achievements in other areas, in particular in naval architecture.

Russell had challenged the wide mathematical community to theoretically

prove existence of the solitary waves he had witnessed and recreated. Further studies of solitary waves were made by Airy (1845) [2] and Stokes (1847) [65]. They both seemed to find it difficult accepting Russell's theories, which seemed to conflict with previous well known and accepted theories of hydrodynamics, set by Newton and Bernoulli.

Boussinesq (1872) and Rayleigh (1876) both also studied the problem and separately found results, suggesting Russell's theories to be true. Indeed whilst the KdV equation (1.2), which was found to model these weakly dispersive waves, appeared in 1895, in fact it was Boussinesq [8] who had derived it earlier. Boussinesq used a fixed co-ordinate system to investigate Russell's wave and ended up with results for the continuity equation and an expression for the wave velocity. It was later on that Korteweg and de Vries derived the KdV equation, using a co-ordinate system moving with the wave [44]. It can actually be seen that substituting Boussinesq's wave velocity into the continuity equation will yield the KdV equation, but at the time this was not done.

However, it wasn't until around the 1960s that the importance of this area of mathematics was found. It was the advent of computer technology and its applications in nonlinear waves, that the KdV equation really was seen to be of great use. In particular it was the field of plasma physics that solitons were found to be of importance. It was thought that the rapid compression of a magnetised plasma by an external magnetic field could maybe produce very high temperatures. Adlam and Allen in 1958 studied the propagation of strong hydromagnetic waves in plasma, but however discovered a solitary wave. This was surprising given that the theory contained no dissipative processes.

In 1965, N. Zabusky and M. Kruskal published a paper which studied numerical solutions to the KdV equation (1.2). In fact it was they who attributed the name of the equation to Korteweg and de Vries, and not to Boussinesq. The equation was known to not only produce the shallow waves that Korteweg and de Vries studied, but also collisionless magnetohydrodynamic waves in plasma. The equation describes waves of finite but limited amplitude. The propagation of a solitary wave was considered and a solution was discovered with its well known properties, such as stronger waves travel more quickly and have a narrower structure. This paper reignited interest in the KdV equation and was responsible for a huge amount of new research into this and related equations. Many developments in varied fields followed this.

More history on the KdV equation can be found in [14, 9]. We also refer to [10], for more on PDEs in the twentieth century, and also [1], for more on solitons, where further references can be found within both.

## 1.2 Similarity Solutions

There are a number of techniques of solving PDEs, including using travelling wave solutions. However similarity solutions are a common way of attempting to understand and occasionally solve some problems. Similarity solutions are used in PDEs, since they simplify the problem by transforming them into ODEs and hence this reduces the number of independent variables. Rescaled variables can be used in studying PDEs, since the coordinate system in which the problem is posed does not affect the formulation of any fundamental physical laws.

Similarity solutions first appeared in *Prandtl's equation*, occurring in *Prandtl's boundary layer theory*, which was proposed in 1904, [54]. This first exact similarity solution in  $\mathbb{R}^2$  is due to Blasius (1908) [6], for the equation of incompressible fluids,

$$\psi_{yyy} + \psi_x \psi_{yy} - \psi_y \psi_{xy} - \psi_{yt} + uu_x + U_t = 0, \quad (1.3)$$

where  $\psi = \psi(x, y, t)$  is the stream function and  $U(x, t)$  is the given external far-field (at  $y = \infty$ ) velocity distribution. It is curious that this is an *odd-order* PDE, though different from those studied here.

### 1.2.1 On Even-Order Models: the Heat Equation

Whilst higher odd-order models have not been studied in great detail, for a number of years various even-order, both lower and higher order, models have been reasonably well understood. The basic linear and most classical case of such a PDE is the *Heat Equation*. It is one of the most important linear partial differential equations and has only recently come to be fairly well understood.

The one-dimensional *Heat Equation* is the canonical PDE

$$u_t = u_{xx} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+. \quad (1.4)$$

It has the classic fundamental solution

$$b(x, t) = \frac{1}{\sqrt{t}} F(y), \quad y = \frac{x}{\sqrt{t}}, \quad (1.5)$$

which takes Dirac's delta as an initial function, i.e., in the sense of distributions,

$$b(x, 0) = \delta(x).$$

Substituting (1.5) into the heat equation (1.4) yields a simple ODE

$$\mathbf{B}F \equiv F'' + \frac{1}{2}(Fy)' = 0, \quad \int F = 1. \quad (1.6)$$

This can be solved to show that the solution,  $F$  is the Gaussian

$$F(y) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}y^2}, \quad (1.7)$$

which is strictly positive and does not have oscillatory components to be traced out for linear dispersion equations. Full spectral theory can also be found, with eigenvalues of the linear second-order operator  $\mathbf{B}$ , given by

$$\lambda_l = -\frac{l}{2}, \quad l = 0, 1, 2, \dots,$$

and eigenfunctions

$$\psi_l = \frac{(-1)^l}{\sqrt{l!}} D_y^l F(y) \equiv H_l(y) F(y).$$

Here  $H_l(y)$  denotes the Hermite polynomials, which are generated by the Gaussian (1.7). These Hermite polynomials were first derived by *C. Sturm* in 1836 [67], where these were used for a classification of all types of multiple spatial zeros for solutions  $u(x, t)$  of the 1D heat equation. This led him to the now famous *Sturm's Theorems on Zero Sets*; see a full history in [21, Ch. 1].

The results found for the linear Heat Equation, have been the basis for other higher and nonlinear even-order models. Whilst solutions to these models retain the basic symmetric Gaussian structure, they have some oscillatory tails instead of a pure simple exponential decay.

### 1.2.2 Spectral theory: Second order Hermite spectral theory from the 19th century, to the 21st century

Spectral theory is an important tool in finding the solutions and behaviour for linear operators. In particular results can be extended and used to describe behaviour for more complicated nonlinear equations.

For the heat equation (1.4), the operator (1.6) is known to be the Hermite self-adjoint case, which follows theories developed in the 19th century. For the higher even-order model

$$u_t = (-1)^{m+1} D_x^{2m} u \quad \text{in } \mathbb{R} \times \mathbb{R},$$

the rescaled operator  $\mathbf{B}_m$  is not self-adjoint, for  $m > 1$ . Here  $m = 1$  corresponds to the heat equation.

For any  $m \geq 1$ , typical scaling

$$b(x, t) = t^{-\frac{1}{2m}} f(y), \quad y = xt^{-\frac{1}{2m}},$$

yields the equation

$$\mathbf{B}_m f \equiv (-1)^{m+1} D_y^{2m} f + \frac{1}{2m} y D_y f + \frac{1}{2m} f = 0, \quad \int_{\mathbb{R}} f(y) dy = 1.$$

The eigenfunctions and their adjoint,  $\{\psi_\beta\}, \{\psi_\beta^*\}$  for the operators  $\mathbf{B}_m$  and  $\mathbf{B}_m^*$  respectively, form an orthonormal basis in the proper weighted space  $L_\rho^2$  (where determination of suitable weights will be carefully explained later), such that the standard bi-orthogonality condition holds:

$$\langle \psi_\beta, \psi_\gamma^* \rangle = \delta_{\beta, \gamma} \quad \text{for all } \beta, \gamma \geq 0.$$

This important property is used in various techniques in describing the behaviour of nonlinear even-order equations. These applications include the semi-linear equation

$$u_t = -(-\Delta)^m u \pm |u|^{p-1} u \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad p > 1, \quad (1.8)$$

and the higher-order Porous Medium Equation (PME)

$$u_t = (-1)^{m+1} \Delta^m (|u|^{n-1} u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad m > 1, \quad n > 1.$$

References to these models may be found later, where we will look at odd-order representations of these generalised equations.

As we will see in Chapter 2, similar odd-order operators are not self-adjoint, even for the lowest third-order order case. Whilst many ideas from related even-

order problems can be applied to the odd-order case, the application is much more difficult and many differences occur. In fact it will be shown that the highly oscillatory nature of the eigenfunctions for the odd-order operator, leaves much of the spectral theory to still remain obscure. In particular it will be seen that dual products between eigenfunctions and their adjoints do not exist in the standard metric, and another metric must be used to calculate these products.

### 1.2.3 Very Singular Solutions

Around the beginning of the 1980s, study of asymptotics of the *semilinear heat equation with absorption*, given by

$$u_t = \Delta u - u^p \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \quad p > 1, \quad (1.9)$$

led to a new class of similarity solutions, called *Very Singular Solutions* (VSS). These are self-similar solutions, which can stable in the evolution and hence, as  $t \rightarrow +\infty$ , attract wide classes of other more general solutions. In addition, as  $t \rightarrow 0$  the solution concentrates at  $x = 0$  with infinite initial mass and this justifies the term “very singular”. We refer to the books [29, 62] for extra details and history concerning VSS. Whilst VSS were being studied early on, it wasn’t until about 1985, in a paper by Kamin and Peletier [35], in which the term VSS was actually used. This was then followed up by Brezis, Peletier and Terman in [11].

The VSS for the semilinear heat equation (1.9) are given by

$$u_*(x, t) = t^{-\frac{1}{p-1}} f(y), \quad y = \frac{x}{\sqrt{t}},$$

where  $f$  solves the elliptic equation

$$\begin{cases} \Delta f + \frac{1}{2}y \cdot \nabla f + \frac{1}{p-1}f - f^p = 0 & \text{in } \mathbb{R}^N, \\ f(y) \text{ has exponential decay as } y \rightarrow \infty. \end{cases}$$

Existence of the VSS was established by Galaktionov, Kurdyumov and Samarskiĭ using a PDE approach in [25] and also by an ODE approach by Brezis, Peletier and Terman, in [11]. Uniqueness of VSS was first proved by Kamin and Véron in [36], by using a PDE comparison method. We also refer to the paper [30], for more references on VSS.

### 1.2.4 The KdV Equation

It is curious that, formally, very singular solutions can be prescribed for the classic KdV equation (1.2). These have a standard self-similar form

$$u_*(x, t) = t^{-\frac{2}{3}} f(y), \quad y = xt^{-\frac{1}{3}}. \quad (1.10)$$

Hence (1.2) reduces to the following ODE:

$$f''' + \frac{1}{3} y f' + \frac{2}{3} f + f f' = 0 \quad \text{in } \mathbb{R}. \quad (1.11)$$

However, whilst we can seemingly construct VSS for the KdV equation, we know that solutions of this type do not exist. The only natural solution we arrive at is the trivial one,  $f \equiv 0$ . This nonexistence conclusion also follows from the global existence results by Kato [38, 37] and Strauss [66].

Indeed, equation (1.2) is invariant under reflection with

$$t \mapsto T - t \quad \text{and} \quad x \mapsto -x, \quad (1.12)$$

where very singular solutions (1.10) are then given by

$$u_*(x, t) = (T - t)^{-\frac{2}{3}} f(y), \quad y = x(T - t)^{-\frac{1}{3}}.$$

Therefore existence of a nontrivial VSS would mean finite time blow-up of solutions, since

$$\sup_x |u(x, t)| \sim (T - t)^{-\frac{2}{3}} \rightarrow +\infty \quad \text{as } t \rightarrow T^-.$$

In view of well-known global existence results for the KdV equation (see references above), this implies the non-existence of very singular solutions for (1.2).

For the generalised KdV (gKdV) equation

$$u_t = u_{xxx} + u^p u_x, \quad (1.13)$$

we can find VSS for  $p \geq 4$ , but not  $p < 4$ . In the case  $p \geq 4$ , (1.13) has self-similar solutions of the standard form

$$u_*(x, t) = t^{-\frac{2}{3p}} f(y), \quad y = xt^{-\frac{1}{3}},$$

where the rescaled kernel satisfies

$$f''' + \frac{1}{3} y f' + \frac{2}{3p} f + f^p f' = 0. \quad (1.14)$$

The critical case  $p = 4$  has been studied in [40], among others, where VSS have conservation and (1.14) can be integrated once, thus establishing existence and uniqueness of the VSS. We do not go through details on why the case  $p = 4$  is critical for VSS solutions of the gKdV equation, but instead refer to the paper by Bona and Weissler, [7].

### 1.3 Three Main Odd-Order Models to Study

We will treat, in particular, three generalised odd-order models, which from our “spectral-like” point of view, essentially have never been looked at before. In particular we look for similarity solutions and using asymptotic, analytic and some numerical methods, we will attempt to find and justify some local and global properties of the rescaled solutions. We use a number of techniques, applied previously for parabolic even-order problems on the odd-order models. However there are difficulties that arise, in particular due to the highly oscillatory nature of fundamental and other solutions, as well as non-symmetry about  $y = 0$ , in the rescaled solutions. Some of these techniques follow classic ideas in functional analysis, which can be found in numerous texts, which include [15, 42, 43, 45, 50, 51]. There are also many books relating to PDEs in specific and some of these include [55, 62].

Our first aim is to study fundamental solutions and develop related spectral theory to the associated rescaled operators, for *linear dispersion* odd-order equations such as

$$u_t = (-1)^{k+1} D_x^{2k+1} u \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad k \geq 1.$$

In doing this, we gain an understanding of its behaviour, which we can then use in the corresponding *semilinear dispersion equation*

$$u_t = (-1)^{k+1} D_x^{2k+1} u - |u|^{p-1} u, \quad \text{where } p > 1.$$

The next goal is to extend this research to PDEs with *nonlinear dispersion*



(the NDEs) of the form

$$u_t = (-1)^{k+1} D_x^{2k+1} (|u|^n u), \quad (1.15)$$

where  $n > 0$  is a fixed parameter. Concerning self-similar solutions of (1.15) (as we call them, “nonlinear eigenfunctions”), asymptotic behaviour and general properties of solutions (existence, uniqueness, shock waves, etc.), very little is known in mathematical literature. First steps of the study of shock and rarefaction waves are performed in [27].

Finally, as a natural extension of the NDE (1.15), we also very briefly discuss the VSS for the NDE with absorption;

$$u_t = (-1)^{k+1} D_x^{2k+1} (|u|^n u) - |u|^{p-1} u.$$

The work on these odd-order models is joint with Prof. V.A. Galaktionov and will form the main part of this thesis.

## 1.4 On Integrable NDEs from Water Wave Theory

Concerning applications of equations associated with nonlinear dispersion operators as in (1.15), it is customary that various odd-order PDEs appear in classic theory of integrable PDEs, such as the classic KdV equation (1.2) or the *fifth-order KdV equation*,

$$u_t + u_{xxxxx} + 30 u^2 u_x + 20 u_x u_{xx} + 10 u u_{xxx} = 0$$

and others from shallow water theory. The quasilinear *Harry Dym equation*

$$u_t = u^3 u_{xx}, \quad (1.16)$$

which also belongs to the NDE family, is one of the most exotic integrable soliton equations; see [28, §4.7] for survey and references therein. Integrable equation theory produced various hierarchies of quasilinear higher-order NDEs, such as

the fifth-order *Kawamoto equation* [39]

$$u_t = u^5 u_{xxxxx} + 5 u^4 u_x u_{xxxx} + 10 u^5 u_{xx} u_{xxx}.$$

Quasilinear integrable extensions are admitted by *Lax's seventh-order KdV equation*

$$\begin{aligned} u_t + \{ 35u^4 + 70[u^2 u_{xx} + u(u_x)^2] \\ + 7[2u u_{xxxx} + 3(u_{xx})^2 + 4u_x u_{xxx}] + u_{xxxxxx} \}_x = 0, \end{aligned}$$

and by the *seventh-order Sawada-Kotara equation*

$$\begin{aligned} u_t + \{ 63u^4 + 63[2u^2 u_{xx} + u(u_x)^2] \\ + 21[u u_{xxxx} + (u_{xx})^2 + u_x u_{xxx}] + u_{xxxxxx} \}_x = 0; \end{aligned}$$

see references in [28, p. 234].

#### Compact pattern formation phenomena and NDEs.

Returning to the lowest third-order NDEs that *are not integrable*, we will also briefly study the *Rosenau-Hyman (RH) equation*

$$u_t = (u^2)_{xxx} + (u^2)_x, \tag{1.17}$$

which has important applications as a widely used model of the effects of nonlinear dispersion, in the pattern formation in liquid drops [59]. It is the  $K(2, 2)$  equation from the general  $K(m, n)$  family of the following NDEs:

$$u_t = (u^n)_{xxx} + (u^m)_x \quad (u \geq 0), \tag{1.18}$$

that also describe various phenomena of compact pattern formation for  $m, n > 1$ , [56, 57]. Such PDEs also appear in curve motion and shortening flows [57]. Similar to well-known parabolic models of porous medium type, the  $K(m, n)$  equation (1.18), with  $n > 1$ , is degenerated at  $u = 0$ , and therefore may exhibit finite speed of propagation and admit solutions with finite interfaces. The crucial advantage of the RH equation (1.17) is that it possesses *explicit* moving compactly supported soliton-type solutions, called *compactons* [59], which are *travelling wave* (TW) solutions.

Various families of quasilinear third-order KdV-type equations can be found

in [13], where further references concerning such PDEs and their exact solutions are given. Higher-order generalised KdV equations are of increasing interest; see e.g., the quintic KdV equation in [32] and [70], where the seventh-order PDEs are studied. For the  $K(2, 2)$  equation (1.17), the compacton solutions were constructed in [56]. More general  $B(m, k)$  equations (indeed, coinciding with the  $K(m, k)$  after scaling)

$$u_t + a(u^m)_x = \mu(u^k)_{xxx},$$

also admit simple semi-compacton solutions [60], as well as the  $Kq(m, \omega)$  nonlinear dispersion equation (another nonlinear extension of the KdV) [56],

$$u_t + (u^m)_x + [u^{1-\omega}(u^\omega u_x)_x]_x = 0.$$

Setting  $m = 2$  and  $\omega = \frac{1}{2}$ , yields a typical quadratic PDE

$$u_t + (u^2)_x + uu_{xxx} + 2u_x u_{xx} = 0,$$

possessing solutions on standard trigonometric-exponential subspaces, where

$$u(x, t) = C_0(t) + C_1(t) \cos lx + C_2(t) \sin lx$$

and  $\{C_0, C_1, C_2\}$  solve a nonlinear 3D dynamical system. Combining the  $K(m, n)$  and  $B(m, k)$  equations gives the dispersive-dissipativity entity  $DD(k, m, n)$  [58],

$$u_t + a(u^m)_x + (u^n)_{xxx} = \mu(u^k)_{xx},$$

that can also admit solutions on invariant subspaces, for some values of parameters.

For the fifth-order NDEs, such as

$$u_t = \alpha(u^2)_{xxxxx} + \beta(u^2)_{xxx} + \gamma(u^2)_x \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad (1.19)$$

compacton solutions were first constructed in [16], where the more general  $K(m, n, p)$  family of PDEs

$$u_t + \beta_1(u^m)_x + \beta_2(u^n)_{xxx} + \beta_3 D_x^5(u^p) = 0 \quad (m, n, p > 1),$$

was introduced. Some of these equations will be treated later on. Equation (1.19)

is also associated with the family  $Q(l, m, n)$  of more general quintic evolution PDEs with nonlinear dispersion,

$$u_t + a(u^{m+1})_x + \omega[u(u^n)_{xx}]_x + \delta[u(u^l)_{xxxx}]_x = 0, \quad (1.20)$$

possessing multi-hump, compact solitary solutions [61].

Concerning higher-order in time quasilinear PDEs, let us mention a generalisation of the *combined dissipative double-dispersive* (CDDD) *equation* (see, e.g., [53])

$$u_{tt} = \alpha u_{xxxx} + \beta u_{xxtt} + \gamma(u^2)_{xxxxt} + \delta(u^2)_{xxt} + \epsilon(u^2)_t \quad (1.21)$$

and also the nonlinear modified dispersive Klein-Gordon equation ( $mKG(1, n, k)$ ),

$$u_{tt} + a(u^n)_{xx} + b(u^k)_{xxx} = 0, \quad n, k > 1 \quad (u \geq 0); \quad (1.22)$$

see some exact TW solutions in [33]. For  $b > 0$ , (1.22) is of hyperbolic (or Boussinesq) type in the class of nonnegative solutions. Let us also mention a related family of 2D *dispersive Boussinesq equations* denoted by  $B(m, n, k, p)$  [69],

$$(u^m)_{tt} + \alpha(u^n)_{xx} + \beta(u^k)_{xxx} + \gamma(u^p)_{yyy} = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}.$$

See [28, Ch. 4-6] for more references and examples of exact solutions on invariant subspaces of NDEs of various types and orders.

# Chapter 2

## Linear Dispersion Equations and Hermitian Spectral Theory

Before looking at some more complicated nonlinear PDEs, it is important to understand how the solutions of linear PDEs behave. The theory formed from the higher-order linear PDEs will be crucial in the understanding of related nonlinear ones. In particular, spectral theory formed in the linear case, will play a large role and will be used in developing understanding of bifurcations, branching and asymptotic behaviour for nonlinear equations.

Thus, we consider the corresponding *linear dispersion equation* (the LDE)

$$u_t = (-1)^{k+1} D_x^{2k+1} u \quad \text{in } \mathbb{R} \times \mathbb{R}_+; \quad k \geq 1. \quad (2.1)$$

Whilst some lower order cases for the odd-order linear PDE (2.1) are generally well understood, the higher-order cases are not. Indeed, it is well-known that the fundamental solution for the case  $k = 1$ , will lead to the classic Airy function.

### 2.1 Fundamental Solutions and Kernels

Consider self-similar fundamental solutions of (2.1), of the form

$$b(x, t) = t^{-\frac{1}{2k+1}} F(y), \quad y = xt^{-\frac{1}{2k+1}}. \quad (2.2)$$

Substituting  $b(x, t)$  into the linear PDE (2.1), we obtain the ODE with respect to  $F$ :

$$\mathbf{B}F \equiv (-1)^{k+1} D_y^{2k+1} F + \frac{1}{2k+1} y D_y F + \frac{1}{2k+1} F = 0, \quad \int F = 1, \quad (2.3)$$

where  $\mathbf{B}$  denotes a key linear operator of the ODE. Note that it is possible to integrate the ODE (2.3) once, to find that  $F(y)$  solves

$$(-1)^{k+1} F^{(2k)} + \frac{1}{2k+1} Fy = 0 \quad \text{for } y \in \mathbb{R}, \quad (2.4)$$

which is now a linear ODE of order  $2k$ . From the original linear PDE of order  $2k + 1$ , the problem has been reduced to a linear ODE of order  $2k$ . Further reductions of this ODE is not possible.

For  $k = 1$ , the solution of this equation yields the classic *Airy function*,

$$F(y) = \text{Ai}(y).$$

However, note that in our case, that due to a difference in sign in the ODE, we actually have  $\text{Ai}(-y)$ , but we shall refer to it as the *Airy function*.

### 2.1.1 Asymptotic Expansion of the Fundamental Kernel

An important technique in trying to find the behaviour of solutions of ODEs is to use asymptotic analysis. This gives the limiting behaviour of solutions, in particular as  $y \rightarrow \pm\infty$ . We refer to the book by Bender and Orszag [5], for various asymptotic techniques. We use here a method of determining the asymptotic behaviour, of the linear ODE, which corresponds to classic WKBJ multi-scale analysis of ODEs, whose basic ideas go back to the 1920s.

Actually, in ODE theory, asymptotics for ODEs such as (2.4) are well-known and have been classified. However, we will need some more refined formulae for further applications and those are not available in standard literature.

Looking at the asymptotic behaviour of the solution of the integrated ODE (2.4), we now write it, for convenience, as

$$(-1)^{k+1} F^{(2k)} = -\frac{1}{2k+1} Fy. \quad (2.5)$$

Let us assume that the rescaled solution  $F(y)$  is of exponential type, as  $y \rightarrow +\infty$ .

In other words, we set for future convenience

$$F(y) = e^{s(y)} \quad \text{as } y \rightarrow +\infty. \quad (2.6)$$

Now assume, as a first approximation, that  $s(y)$  is some polynomial, such that  $s(y) \sim ay^b$ . Then the kernel  $F(y)$  and its derivatives may be given by

$$\begin{aligned} F' &= s' e^s, \\ F'' &= [s'' + (s')^2] e^s, \\ F''' &= [s''' + 3s's'' + (s')^3] e^s, \\ &\vdots \\ F^{(2k)} &= [s^{(2k)} + \dots + k(2k-1)(s')^{2k-2}s'' + (s')^{2k}] e^s, \end{aligned}$$

where  $s' \sim aby^{b-1}$ ,  $s'' \sim ab(b-1)y^{b-2}$ ,  $\dots$ ,  $s^{(2k)} \sim \frac{ab!}{(b-2k)!}y^{b-2k}$ . Substituting this into the ODE (2.5), it can easily be seen that all the  $e^{s(y)}$  terms cancel, due to the linearity of the equation. From the resulting equation, we look to do a dominant balance analysis, in order to determine the leading order behaviour of  $F(y)$ . In doing so, we find a first approximation for the function  $s(y)$ . The balance of the equation depends on the value of the parameter  $b$  and we obtain two different cases.

If  $b \leq 0$ , then as  $y \rightarrow +\infty$ , every term on the LHS of the ODE is  $o(y)$ . Therefore there is no balance in this case.

If  $b > 0$ , then  $y^{b-\tilde{n}} = o(y^{2k(b-1)})$ , for any  $\tilde{n} > 1$ . Therefore balancing leading terms, we have that

$$(-1)^{k+1}a^{2k}b^{2k}y^{2k(b-1)} \sim -\frac{1}{2k+1}y,$$

for all  $k \in \mathbb{Z}_+$ . By first equating powers of  $y$ , and then coefficients, we find our parameters

$$b = \frac{2k+1}{2k}$$

and

$$a = -i(2k+1)^{-\frac{1}{2k}} \left( \frac{2k}{2k+1} \right).$$

Hence we now have the first approximation to  $s(y)$ , with

$$\begin{aligned} s(y) &= -2ki \left( \frac{y}{2k+1} \right)^{\frac{2k+1}{2k}} + c(y), \\ s'(y) &= -i \left( \frac{y}{2k+1} \right)^{\frac{1}{2k}} + c'(y), \\ s''(y) &= -\frac{i}{2k(2k+1)^{\frac{1}{2k}}} y^{-\frac{2k-1}{2k}} + c''(y), \\ &\dots \end{aligned}$$

Here  $c(y) \sim o(y^{\frac{2k+1}{2k}})$  is some function of  $y$  and the next term in the approximation of  $s(y)$ . Since  $k \geq 1$ , we must have that  $c^{(m+1)}(y) \sim o(c^{(m)}(y))$  for any  $m > 1$ , which will be used in determining leading order terms.

We attempt to find this function  $c(y)$ , in order to improve the approximation of  $s(y)$ . We let  $g(y) = -2ki \left( \frac{y}{2k+1} \right)^{\frac{2k+1}{2k}}$  for convenience, to see how the terms are balanced. Hence balancing the leading order terms yields

$$c'(y) \sim -\frac{1}{2} (2k-1) \frac{g''(y)}{g'(y)} = -\frac{2k-1}{4ky}.$$

Integrating this, the next term in the expansion can be found to be

$$c(y) \sim -\frac{2k-1}{4k} \ln y.$$

The second term in this expansion,  $c(y)$ , is called the *controlling factor*. It is an important term, as we will see that it governs the decay (or any possible growth) of solutions as  $y \rightarrow +\infty$ .

Whilst with the first two terms, we can have a clear idea of the leading order behaviour, it is possible to find a better approximation with further expansions. In this case, we look to see if any lower terms affect the exponential and so now expand once again with

$$s(y) = -2ki \left( \frac{y}{2k+1} \right)^{\frac{2k+1}{2k}} - \frac{2k-1}{4k} \ln y + d(y),$$

where  $d(y) = o(c(y))$ . Balancing leading order terms once again, we find that

$$d'(y) \sim -\frac{2k-1}{4k^2 y^2}.$$



Hence by integrating, it is found that the third term in the expansion is given by,

$$d(y) \sim \frac{2k-1}{4k^2 y}.$$

Note that this is the third term in the expansion and hence all following lower order terms are  $o(1)$  and so by (2.6), this does not affect the exponential.

From this, we find that the asymptotic behaviour of  $f(y)$ , can be given by

$$F(y) \sim y^{-\frac{(2k-1)}{4k}} \exp\left(-2ki\left(\frac{y}{2k+1}\right)^{\frac{2k+1}{2k}}\right) \quad \text{as } y \rightarrow +\infty.$$

Therefore, for real solutions, we have that

$$F(y) \sim y^{-\frac{(2k-1)}{4k}} \cos\left(d_k y^{\frac{2k+1}{2k}} + \hat{c}\right) \quad \text{as } y \rightarrow +\infty, \quad (2.7)$$

where

$$d_k = 2k \left(\frac{1}{2k+1}\right)^{\frac{2k+1}{2k}}, \quad (2.8)$$

and  $\hat{c}$  is some constant.

The same analysis can be applied for  $y \rightarrow -\infty$ , by letting  $y \mapsto -y$  and performing the same calculations. Hence, we find that the rescaled kernel decays exponentially fast in the opposite direction,

$$\begin{aligned} F(y) &\sim |y|^{-\frac{(2k-1)}{4k}} \cos\left(d_k |y|^{\frac{2k+1}{2k}} \sin b_k + \hat{c}\right) \\ &\times \exp\left(d_k |y|^{\frac{2k+1}{2k}} \cos b_k\right) \quad \text{as } y \rightarrow -\infty, \end{aligned}$$

where  $d_k$  is as before and

$$b_k = \begin{cases} \frac{\pi}{k} \lfloor \frac{k+1}{2} \rfloor & \text{for even } k, \\ \frac{\pi}{k} \left(\frac{k+1}{2}\right) + \frac{\pi}{2k} & \text{for odd } k. \end{cases}$$

We note that the coefficient  $a$  has many roots, which represent different solutions of the ODE. However, we only want roots such that there is exponential decay, rather than growth. Therefore we exclude the roots where there is growth, which corresponds to  $\text{Re } a > 0$ . The asymptotics here show the behaviour of the first roots, such that  $\text{Re } a \leq 0$ . The method of finding these roots and all other roots may be seen in Section 2.5, which also explains more carefully which roots we need to look at.

As  $y \rightarrow +\infty$ , we have a complex part for all values of  $k$  and this gives slow decaying oscillatory behaviour. The decay of the oscillations also increases for larger  $k$ . As  $y \rightarrow -\infty$ , we have oscillations for  $k > 1$  and these oscillations are always exponentially small. For the special case of  $k = 1$ , we note that  $b_k = \pi$ , hence  $\sin b_k = 0$ . This gives the pure exponential decaying behaviour

$$F(y) \sim |y|^{-\frac{1}{4}} e^{-2 \cdot (\frac{1}{3}|y|)^{\frac{3}{2}}} \quad \text{as } y \rightarrow -\infty.$$

This is the reason why the Airy function is the only odd-order linear case where there is no exponential oscillatory behaviour, as  $y \rightarrow -\infty$ .

We now summarise the results of our asymptotic analysis:

**Proposition 2.1.1** *The rescaled kernel  $F(y)$ , of the fundamental solution (2.2), for the linear PDE (2.1), satisfies*

$$|F(y)| \leq \begin{cases} D_0(1+y^2)^{-\frac{(2k-1)}{8k}} e^{-d_k|y|^\alpha} & \text{for } y \leq 0, \\ D_0(1+y^2)^{-\frac{(2k-1)}{8k}} & \text{for } y \geq 0, \end{cases} \quad (2.9)$$

where  $D_0$  is a positive constant dependent on  $k$  and

$$\alpha = \frac{2k+1}{2k} \in (1, 2) \quad \text{for all } k \geq 1.$$

## 2.1.2 Numerical Construction of Fundamental Kernels

Numerics play an important part in the theory of differential equations. They are an important way of looking at the behaviour of the solutions and to check any results found. Results in this section were generated using the Matlab bvp4c solver, to look at the singular solutions of the linear ODE (2.3).

Let  $F$  be a solution of (2.4), then  $cF$  is also a solution, for all  $c \in \mathbb{R}$ . Due to such “non-uniqueness” (and to some extent “instability”) of the solutions, the zero solution is likely to be found using numerical methods. Therefore in order to ensure that a non-zero solution is obtained, we set the (normalisation) constraint

$$\max |F| = 1, \quad (2.10)$$

which is attained at some point  $y = \hat{a}$ . We then solved the ODE for different right and left solutions, at this maximum point  $\hat{a}$ , using the BVP solver. As the

ODE is of second order (for the case  $k = 1$ ), each computation must have two boundary conditions placed. Since  $\hat{a}$  is fixed as a maximum, the first derivative is also zero at this point. The other boundary condition placed was to ensure that  $F(y) = 0$ , at an end point that is sufficiently removed from  $\hat{a}$ . In order for a match of right and left solutions at the point  $\hat{a}$ , we needed to have the second derivative of the solution to be the same here, in order for  $F(y)$  to be continuous at  $y = \hat{a}$ . So the value of  $\hat{a}$  was moved in order to match the second derivative for the right and left solutions. See Figure 2-1, which was obtained by this shooting method.

A similar method was applied to the fifth order ( $k = 2$ ) equation, where values of the left-hand solution were used as boundary conditions for the right solution, and the correct value for  $\hat{a}$  was found by matching the fourth derivative. See Figure 2-2.

These boundary conditions follow on from the asymptotic analysis. For  $y \rightarrow -\infty$ , there is fast exponential decay, which ensures the solution will be zero at some  $y \ll 0$ . Similarly, for  $y \rightarrow +\infty$ , the behaviour is oscillatory and hence there are infinitely many points at which  $F(y) = 0$ .

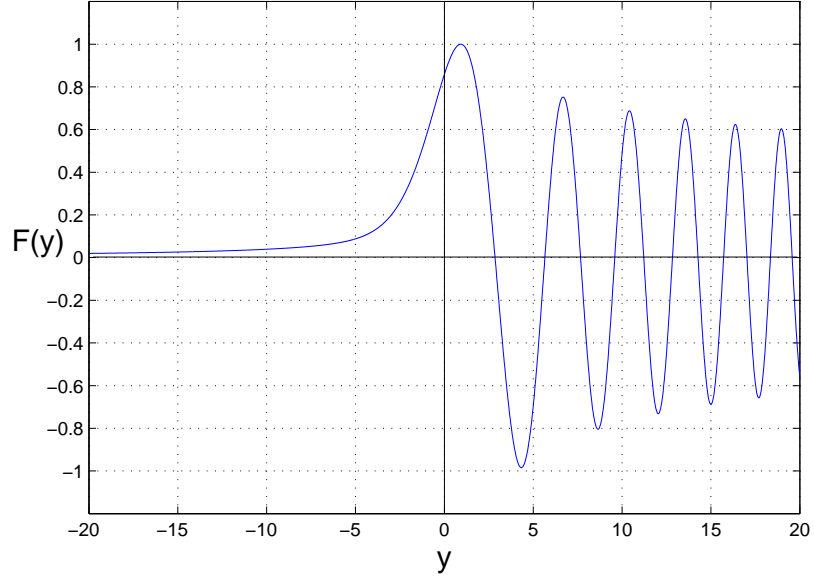
Obviously due to the method used, we cannot guarantee that exact solutions were found, but these show the behaviour of the solutions. Also note that the plots do not show the fundamental kernels of the ODE, such that  $\int F = 1$ , but rescaled profiles such that (2.10) is satisfied. It is noted that there are most likely other methods, in which the rescaled solutions may be found. However, this method overcomes difficulties where there are infinitely many solutions and may be used for similar problems, when the behaviour is known. In particular this method can be used for higher order equations, where finding correct shooting parameters is more difficult.

As one can see, from comparing Figures 2-1 and 2-2, the oscillations are smaller as  $k$  increases. This follows from the asymptotic analysis done in Section 2.1.1. Decay is very slow as  $y \rightarrow +\infty$  also, especially for the *Airy* function, which corresponds to  $k = 1$ . There are also traces of the exponentially small oscillatory behaviour, as  $y \rightarrow -\infty$ , in the case  $k = 2$ ; see Figure 2-2.

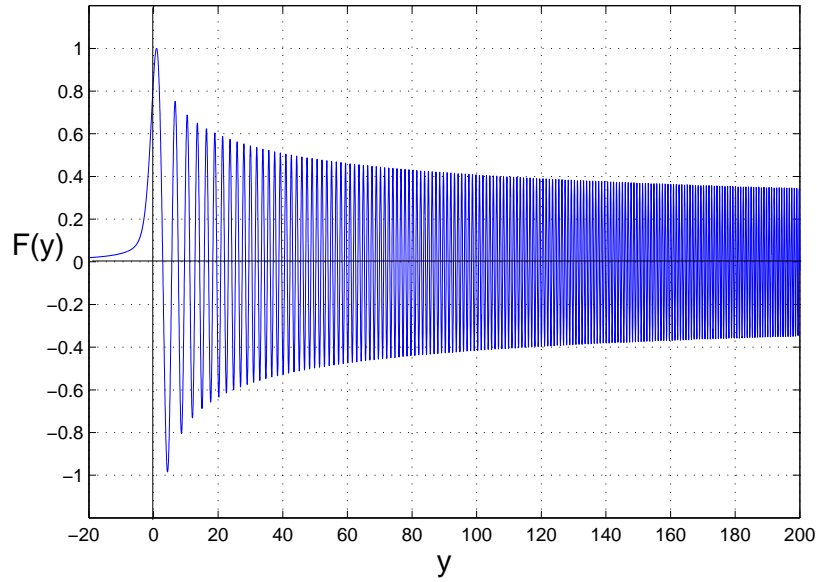
For convenience we denote these rescaled kernels as higher order *Airy* functions by

$$F(y) = \text{Ai}_{2k+1}(y) \quad \text{for } k = 1, 2, \dots, \quad (2.11)$$

so that  $\text{Ai} = \text{Ai}_3$ .



(a) Oscillations of the solution.



(b) "Tail" of the solution, for  $y \gg 1$ .

Figure 2-1: The rescaled kernel  $F(y) = \text{Ai}(y)$ , of the fundamental solution (2.2), for  $k = 1$ , to  $u_t = u_{xxx}$ .

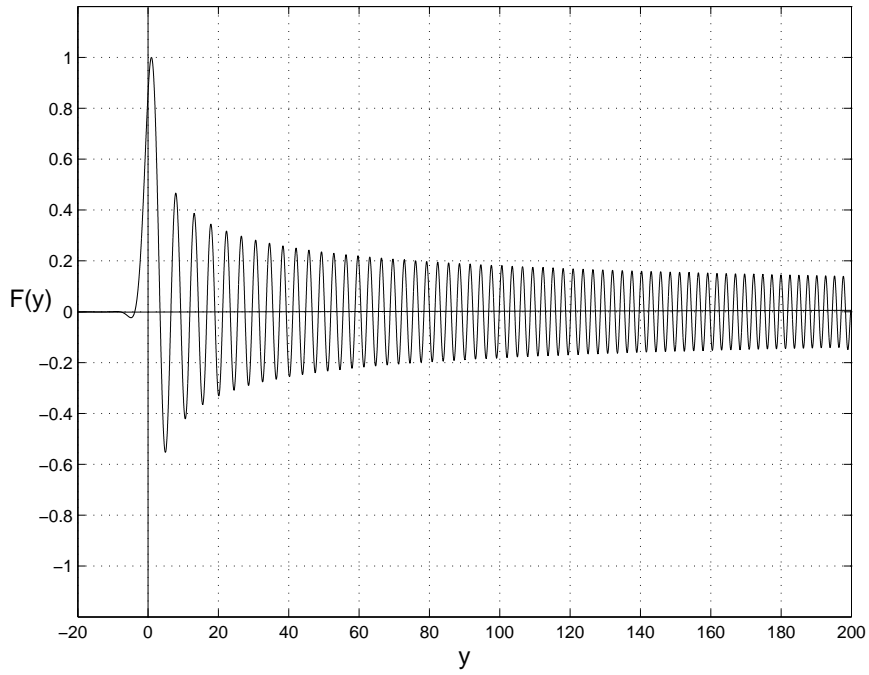


Figure 2-2: The rescaled kernel  $F(y)$ , of the fundamental solution (2.2), for  $k = 2$ , to  $u_t = -u_{xxxxx}$ . We denote this as  $\text{Ai}_5(y)$ .

## 2.2 Explicit Semigroup Representations

Whilst behaviour of the rescaled kernels for the ODE (2.3), are characterised by the asymptotics in Section 2.1.1, and for two lower order cases in Section 2.1.2, in order to build an understanding of related nonlinear equations, spectral theory for the LDEs must be formed. We refer to books on semigroup theory [48, 4], which will be used to find the spectrum of the linear operator,  $\mathbf{B}$ .

### 2.2.1 Operator $\mathbf{B}$

Let  $u(x, t)$  be the solution of the Cauchy problem, for the linear PDE (2.1), with bounded integrable initial data  $u(x, 0) \equiv u_0(x)$ . The solution is then represented by the convolution of initial data with the fundamental solution:

$$u(x, t) = b(t) * u_0 \equiv t^{-\frac{1}{2k+1}} \int_{\mathbb{R}} F((x - z)t^{-\frac{1}{2k+1}}) u_0(z) \, dz. \quad (2.12)$$

Let us now introduce further rescaling of variables, corresponding to the variables of the fundamental solution (2.2),

$$u(x, t) = t^{-\frac{1}{2k+1}} w(y, \tau), \quad y = xt^{-\frac{1}{2k+1}}, \quad \tau = \ln t : \mathbb{R}_+ \rightarrow \mathbb{R},$$

where we now have scaling with respect to time as well. The rescaled solution  $w(y, \tau)$  then satisfies the evolution equation

$$w_\tau = \mathbf{B}w, \tag{2.13}$$

where  $\mathbf{B}$  is the linear operator for the rescaled kernel described in (2.3). More precisely,  $\mathbf{B}$  is still a linear differential expression to be equipped with proper “boundary conditions”.

Here  $w(y, \tau)$  satisfies the Cauchy problem for (2.13) in  $\mathbb{R} \times \mathbb{R}_+$ , with initial data at  $\tau = 0$  (i.e., at  $t = 1$  and not  $t = 0$ ). So now the initial data are given by

$$w_0(y) = u(y, 1) \equiv b(1) * u_0 = F * u_0. \tag{2.14}$$

Hence the linear operator  $\frac{\partial}{\partial \tau} - \mathbf{B}$  is the rescaled version of the original linear dispersion operator

$$\frac{\partial}{\partial t} + (-1)^k D_x^{2k+1}.$$

Therefore the corresponding semigroup  $e^{\mathbf{B}\tau}$  admits an explicit integral representation. This helps to establish some properties of  $\mathbf{B}$ , including key spectral ones and describe other evolution features of the linear flow.

Rescaling convolution (2.12) gives the explicit representation of the semigroup

$$w(y, \tau) = \int_{\mathbb{R}} F(y - ze^{-\frac{\tau}{2k+1}}) u_0(z) dz \equiv e^{\mathbf{B}\tau} w_0 \quad \text{for } \tau \geq 0.$$

For any  $y \in \mathbb{R}$ , Taylor’s Power Series, for the analytic kernel  $F$ , can be used to expand the convolution and obtain

$$\begin{aligned} F(y - ze^{-\frac{\tau}{2k+1}}) &= \sum_{(\beta)} e^{-\frac{|\beta|\tau}{2k+1}} \frac{(-1)^{|\beta|}}{\beta!} D_y^\beta F(y) z^\beta \\ &\equiv \sum_{(\beta)} e^{-\frac{|\beta|\tau}{2k+1}} \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} \psi_\beta(y) z^\beta. \end{aligned}$$

Here, for the first time, we introduce the *eigenfunctions* of the linear operator

**B** as

$$\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta}!} D_y^\beta F(y), \quad \beta \geq 0. \quad (2.15)$$

We note that whilst  $|\beta| \equiv \beta$  here, this is not true in the general case of  $\mathbb{R}^N$ , where  $\beta$  stands for a multi-index. However the notation of  $|\beta|$  is used, to show that the theory here may be extended to multi-dimensional spaces. We bear in mind, that for the typical linear dispersion operator in  $\mathbb{R}^N$ ,

$$\mathbf{B}F = (-1)^{k+1} \frac{\partial}{\partial y_1} \Delta^k F + \frac{1}{2k+1} y \cdot \nabla_y F + \frac{N}{2k+1} F, \quad (2.16)$$

where it appears after similar scaling of the following LDE:

$$u_t = (-1)^k \frac{\partial}{\partial x_1} \Delta^k u. \quad (2.17)$$

Here the multi-index will be given by  $\beta = (\beta_1, \dots, \beta_N)$ , with the length  $|\beta| = \beta_1 + \dots + \beta_N$ . With this notation, some of our basic results can be directly translated to operators such as (2.16) in  $\mathbb{R}^N$ . From now onwards, this notation is not used.

Looking back at the expansion of the convolution, the solution of (2.13) can be represented by

$$w(y, \tau) = \sum_{(\beta)} e^{-\frac{\beta\tau}{2k+1}} M_\beta(u_0) \psi_\beta(y), \quad (2.18)$$

where thus  $\lambda_\beta = -\frac{\beta}{2k+1}$  and  $\psi_\beta(y)$  are the eigenvalues and eigenfunctions of the operator **B** and

$$M_\beta(u_0) = \frac{1}{\sqrt{\beta}!} \int_{\mathbb{R}} z^\beta u_0(z) dz. \quad (2.19)$$

Here  $M_\beta(u_0)$  are the corresponding moments of the initial data  $w_0$ , i.e., “scalar products” of  $w_0$  with some “adjoint polynomials”, to be detected shortly, together with a proper metric involved.

We now introduce an equivalent explicit representation of the semigroup for **B**, which more clearly determine the eigenfunctions of the adjoint operator **B**<sup>\*</sup>, to be introduced and studied in the next subsection. We perform another rescaling to exclude the relation (2.14) in order to find the correct semigroup, corresponding to the initial data at  $t = 0$ :

$$u = (1+t)^{-\frac{1}{2k+1}} w, \quad y = x(1+t)^{-\frac{1}{2k+1}}, \quad \tau = \ln(1+t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+.$$

Then rescaling the convolution gives

$$\begin{aligned} w(y, \tau) &= e^{\mathbf{B}\tau} u_0 \\ &\equiv (1 - e^{-\tau})^{-\frac{1}{2k+1}} \int_{\mathbb{R}} F\left((y - ze^{-\frac{\tau}{2k+1}})(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) w_0(z) dz. \end{aligned} \quad (2.20)$$

Once again we look to find explicit representations for the eigenfunctions, eigenvalues and adjoint eigenfunctions, given in the dual products  $\langle u_0, \psi_\beta^* \rangle$ , in the standard metric of  $L^2$ . It will be shown later that we can actually determine the adjoint eigenfunctions,  $\{\psi_\beta^*\}$ , using a much easier method.

Looking at our rescaled equation (2.20), by Taylor's expansion we have

$$\begin{aligned} &F\left((y - ze^{-\frac{\tau}{2k+1}})(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) \\ &= \sum_{(\mu)} \frac{(-1)^\mu}{\mu!} D_y^\mu F\left(y(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) (e^\tau - 1)^{-\frac{\mu}{2k+1}} z^\mu \end{aligned}$$

and

$$F\left(y(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) = \sum_{(\nu)} \frac{1}{\nu!} D_y^\nu(F)(0) y^\nu (1 - e^{-\tau})^{-\frac{\nu}{2k+1}}.$$

Then using these expansions, our solution is given by

$$\begin{aligned} w(y, \tau) &= (1 - e^{-\tau})^{-\frac{1}{2k+1}} \sum_{(\mu, \nu)} \frac{(-1)^\mu}{\mu! \nu!} D_y^\nu(F)(0) D_y^\mu(y^\nu) (e^\tau - 1)^{-\frac{\mu}{2k+1}} \\ &\quad \times (1 - e^{-\tau})^{-\frac{\nu}{2k+1}} \int_{\mathbb{R}} z^\mu w_0(z) dz. \end{aligned} \quad (2.21)$$

Rearranging this we have

$$\begin{aligned} w(y, \tau) &= \sum_{(\mu, \nu)} e^{-\frac{\mu\tau}{2k+1}} (1 - e^{-\tau})^{-\frac{\mu+\nu+1}{2k+1}} \frac{(-1)^\mu}{\mu! \nu!} D_y^\nu(F)(0) D_y^\mu(y^\nu) \\ &\quad \times \int_{\mathbb{R}} z^\mu w_0(z) dz. \end{aligned}$$

Let us now expand the term  $(1 - e^{-\tau})^{-\frac{\mu+\nu+1}{2k+1}}$  in  $e^{-\tau}$ , so that the rescaled solution may be represented by

$$\begin{aligned} w(y, \tau) &= \sum_{(\mu, \nu, \phi)} e^{-\frac{\mu\tau}{2k+1}} \frac{(-1)^\phi}{\phi!} \left(-\frac{\mu+\nu+1}{2k+1}\right) \dots \left(-\frac{\mu+\nu+1}{2k+1} - \phi + 1\right) e^{-\phi\tau} \frac{(-1)^\mu}{\mu! \nu!} \\ &\quad \times D_y^\nu(F)(0) D_y^\mu(y^\nu) \int_{\mathbb{R}} z^\mu w_0(z) dz. \end{aligned} \quad (2.22)$$



We note that this is now a summation of three parameters, which is a difficult equation to analyse.

We first look to find the semigroup of the operator  $\mathbf{B}$ , which we see can be given by

$$e^{\lambda_\beta \tau} \equiv e^{-\left(\frac{\mu}{2k+1} + \phi\right)\tau}.$$

In order to find the representation of our semigroup, we compare with the previous scaling, given by (2.18), so that

$$\lambda_\beta = -\frac{\beta}{2k+1} \implies \beta = \mu + (2k+1)\phi.$$

From this it can easily be seen that the parameter  $\phi$  may be given in terms of  $\mu$  and  $\beta$ , with

$$\phi = \frac{\beta - \mu}{2k+1}.$$

However  $\phi > 0$  and so  $\mu \leq \beta$ . Hence since the spectrum of  $\mathbf{B}$  is discrete (in a certain proper functional setting to be fixed), then  $\beta$  is fixed and finite, and so  $\mu$  must also be finite.

This gives the representation of our semigroup, for the rescaled solution, which corresponds to the correct initial data, for  $w_0(y) = u_0(x)$ . The remaining terms in the expansion will give rise to the polynomial eigenfunctions and dual product between the adjoint eigenfunctions and initial data, such that

$$w(y, \tau) = \sum_{(\beta)} e^{\lambda_\beta \tau} \psi_\beta(y) \langle u_0, \psi_\beta^*(z) \rangle.$$

For our expansion, we so far have reduced to

$$\begin{aligned} w(y, \tau) = & \sum_{(\mu, \nu, \phi)} e^{\lambda_\beta \tau} \frac{(-1)^\phi}{\phi!} \left( -\frac{\mu+\nu+1}{2k+1} \right) \dots \left( -\frac{\mu+\nu+1}{2k+1} - \phi + 1 \right) \frac{(-1)^\mu}{\mu! \nu!} \\ & \times D_y^\nu(F)(0) D_y^\mu(y^\nu) \int_{\mathbb{R}} z^\mu w_0(z) dz. \end{aligned} \tag{2.23}$$

We note that we can express the terms

$$\frac{(-1)^\mu}{\mu!} D_y^\mu \left( \sum_{(\nu)} \frac{1}{\nu!} D^\nu F(0) y^\nu \right) = \frac{(-1)^\mu}{\mu!} D_y^\mu F(y).$$

It is first noted that the summation over  $\nu$  must be an infinite sum, for the expression to give us  $F(y)$ . However we have that  $\nu$  is represented in the coefficients

and so this cannot be the case. Also for this term to be exactly  $\psi_\beta$ , then we must have that  $\mu = \beta$ , which is also not true, since this means we have that the index  $\phi = 0$ .

The term  $\int z^\mu dz$ , also gives rise to the finite polynomial adjoint eigenfunctions,  $\psi_\beta^*(y)$  (called *generalised Hermite polynomials*, which are extensions of classical ones mentioned at the beginning). These must also be finite since we are summing over  $\mu$ , which we have already established is finite, so some open questions remain. In other words, calculating finite adjoint polynomials from (2.23) is not easy at all, and is even questionable.

As stated before, we will obtain these polynomials more explicitly, using a much easier method based on the differential expression for  $\mathbf{B}^*$ .

The moments  $\int_{\mathbb{R}} z^\mu w_0 dz$  are finite for all continuous data  $w_0$  with sufficient decay at infinity (say, with compact support). Whilst due to the expansion of terms using Taylor's Series, it suggests an infinite sum, we expect that the interaction between the summations will give us a finite sum.

Thus, for our linear PDE, we have studied the asymptotic behaviour for the problem

$$\begin{cases} u_t = (-1)^{k+1} D_x^{2k+1} u, & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u_0 \in L_\rho^2, \quad \rho = e^{a|y|^{\frac{2k+1}{2k}}}, \end{cases} \quad (2.24)$$

with initial data  $u_0$  and where the weight of the space,  $\rho$  and the constant  $a > 0$  are to be properly defined later on.

For the multi-dimensional model (2.17), we use the notation  $l = |\beta|$ . In general for the present one-dimensional case, we shall use both, where  $l \equiv \beta$ .

**Theorem 2.2.1** *For (2.24), the eigenfunction expansion of the semigroup (2.18) implies that  $\forall u_0 \in L_\rho^2$ , there exists a finite  $l$  such that, as  $t \rightarrow +\infty$ ,*

$$u(x, t) = t^{-\frac{1}{2k+1} - \lambda_l} [c_l \psi_l(xt^{-\frac{1}{2k+1}}) + o(1)],$$

where  $l = \beta$  is the first eigenvalue index, for which the corresponding moment  $M_\beta(u_0) \neq 0$ .

The following uniqueness conclusion is straightforward and we keep this as a simple illustration for further results:

**Corollary 2.2.2** *Assume that, for any  $k \geq 1$ , the solution of (2.24) satisfies*

$$\sup_x |u(x, t)| = o(t^{-k}) \quad \text{as } t \rightarrow \infty.$$

*Then  $u(x, t) \equiv 0$ .*

Such results belong to Carleman–Agmon-type estimates in operator theory: if a solution of a linear equation, under proper conditions on operators involved, decays super-exponentially fast (in terms of  $\tau = \ln t$ ) as  $t \rightarrow +\infty$ , then it is trivial.

Thus, we conclude with the following suggestions:

- There exists point spectrum  $\{\lambda_\beta = -\frac{\beta}{2k+1}, \beta \geq 0\}$  of the non self-adjoint operator  $\mathbf{B}$ .
- We have no integral terms in the expansion, hence the spectrum is expected to be discrete.
- The set of eigenfunctions  $\{\psi_\beta\}$  seems to be complete and closed in the corresponding weighted  $L^2$ -space.
- Traces of the polynomials, which give rise to the adjoint eigenfunctions,  $\{\psi_\beta^*\}$ .

### 2.2.2 Semigroup of the Adjoint Operator, $\mathbf{B}^*$

We now find the explicit representation of the semigroup  $e^{\mathbf{B}^*\tau}$ , where  $\mathbf{B}^*$  is obtained from the LDE in (2.24) by using other *blow-up* rescaling. Let us introduce the rescaled variables

$$u(x, t) = w(y, \tau), \quad y = x(1 - t)^{-\frac{1}{2k+1}}, \quad \tau = -\ln(1 - t) : (0, 1) \rightarrow \mathbb{R}_+.$$

Then  $w(y, \tau)$  now solves the problem

$$w_\tau = \mathbf{B}^* w \quad \text{for } \tau > 0,$$

where  $w(y, 0) = w_0(y) = u_0(x)$ . Here the adjoint operator  $\mathbf{B}^*$  is given by

$$\mathbf{B}^* = (-1)^{k+1} D_y^{2k+1} - \frac{1}{2k+1} y D_y.$$

By rescaling the convolution (2.12), we have

$$w(y, \tau) = (1 - e^{-\tau})^{-\frac{1}{2k+1}} \int_{\mathbb{R}} F\left((ye^{-\frac{\tau}{2k+1}} - z)(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) w_0(z) dz.$$

Using Taylor's expansion yields

$$\begin{aligned} & F\left((ye^{-\frac{\tau}{2k+1}} - z)(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) \\ &= \sum_{(\beta)} \frac{(-1)^\beta}{\beta!} (1 - e^{-\tau})^{-\frac{\beta}{2k+1}} D^\beta F\left(ye^{-\frac{\tau}{(2k+1)}}(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) z^\beta \end{aligned}$$

and expanding in  $y$  leads to

$$F\left(ye^{-\frac{\tau}{2k+1}}(1 - e^{-\tau})^{-\frac{1}{2k+1}}\right) = \sum_{(\nu)} \frac{1}{\nu!} D^\nu F(0) y^\nu (e^\tau - 1)^{-\frac{\nu}{2k+1}}.$$

So the solution may be represented by

$$\begin{aligned} w(y, \tau) &= (1 - e^{-\tau})^{-\frac{1}{2k+1}} \sum_{(\beta, \nu)} \frac{(-1)^\beta}{\beta! \nu!} (1 - e^{-\tau})^{-\frac{\beta}{2k+1}} (e^\tau - 1)^{-\frac{\nu}{2k+1}} \\ &\quad \times D^\nu F(0) \frac{1}{(\nu - \beta)!} y^{\nu - \mu} \int_{\mathbb{R}} z^\beta w_0(z) dz. \end{aligned} \tag{2.25}$$

Again, similar to  $\mathbf{B}$  in the previous subsection, (2.25) can be viewed as an eigenfunction expansion of the solution that can reveal many key spectral properties of  $\mathbf{B}^*$ . However, further refining of this expansion will lead to more complicated formulae, which are not that effective and useful. Therefore, we return later on to polynomial eigenfunctions of  $\mathbf{B}^*$  on the basis of a simpler direct approach.

## 2.3 Hermitian Spectral Theory: Discrete Point Spectrum of the Operator $\mathbf{B}$

We now start more systematically to develop necessary spectral theory for the operator pair  $\{\mathbf{B}, \mathbf{B}^*\}$ , introduced above. In some aspects, this theory repeats classic steps of self-adjoint theory for the classic Hermite operator (1.6), which since the nineteenth century, is associated with the name of Sturm and other outstanding mathematicians.

We calculate the spectrum of the linear operator  $\mathbf{B}$ , in the weighted space

$L^2_\rho(\mathbb{R})$ , with the exponential weight

$$\rho(y) = \begin{cases} e^{a|y|^\alpha} & \text{for } y \leq -1, \\ e^{-ay^\alpha} & \text{for } y \geq 1, \end{cases}$$

where we define  $\rho(y)$  to be sufficiently smooth in the complete interval  $(1, -1)$ . Here we have that  $\rho(y) > 0$  and  $a \in (0, 2d)$ , is a sufficiently small positive constant, where  $d$  is to be defined later in Section 2.5.3. The power  $\alpha$ , is as defined before and given by

$$\alpha = \frac{2k+1}{2k} \in (1, 2) \quad \text{for all } k \geq 1.$$

We introduce a Hilbert space of functions  $H_\rho^{2k+1}(\mathbb{R})$  with the inner product

$$\langle v, w \rangle_{2k+1, \rho} = \int_{\mathbb{R}} \rho(y) \sum_{r=0}^{2k+1} D_y^r v(y) D_y^r w \, dy$$

and therefore the induced norm is

$$\|v\|_{2k+1, \rho}^2 = \int_{\mathbb{R}} \rho(y) \sum_{r=0}^{2k+1} |D_y^r v(y)|^2 \, dy.$$

So we have that  $H_\rho^{2k+1} \subset L_\rho^2(\mathbb{R}) \subset L^2(\mathbb{R})$ .

**Lemma 2.3.1**  *$\mathbf{B}$  is a bounded linear operator from  $H_\rho^{2k+1}(\mathbb{R})$  to  $L_\rho^2(\mathbb{R})$ .*

**Proof.** We look at the linear operator given by (2.3),

$$\mathbf{B}v = (-1)^{k+1} v^{(2k+1)} + \frac{1}{2k+1} y v' + \frac{1}{2k+1} v.$$

For  $\mathbf{B}$  to be bounded, it is necessary to look at the second term with the unbounded coefficient  $y$ . In order to do this, we want to show that

$$\int \rho(y v')^2 \leq C \int \rho(v^{(2k+1)})^2 \, dy,$$

for some constant  $C > 0$ . To show this, we look at the non-negative integral

$$\begin{aligned} 0 &\leq \int \rho(v' + y^\gamma v)^2 dy \\ &= \int \rho((v')^2 + y^{2\gamma} v^2 + 2y^\gamma v' v) dy, \end{aligned}$$

where  $\gamma > 0$  is some unknown exponential. Integrating by parts in the last term, we have that

$$\begin{aligned} 2 \int (\rho y^\gamma) v' v dy &= \int (\rho y^\gamma) (v^2)' dy \\ &= - \int v^2 (\rho y^\gamma)' dy. \end{aligned}$$

We can integrate those absolutely convergent indefinite integrals by parts, for functions in the spaces  $H_\rho^{2k+1}(\mathbb{R})$  and  $L_\rho^2(\mathbb{R})$ .

Then for exponential weight

$$\rho(y) = e^{-ay^\alpha},$$

we have that

$$(\rho y^\gamma)' \sim e^{-ay^\alpha} y^{\gamma+\alpha-1},$$

for  $y \gg 1$  and also for  $y \ll -1$ , where we have to replace  $y \mapsto |y|$ . Hence

$$\int \rho(v')^2 dy + \int \rho y^{2\gamma} v^2 dy + C_1 \int \rho y^{\gamma+\alpha-1} v^2 dy \geq 0.$$

By equating powers of  $y$ , this yields

$$\gamma = \alpha - 1.$$

Substituting for  $\gamma$  we have that

$$\int \rho y^{2(\alpha-1)} v^2 dy \leq C_2 \int \rho(v')^2 dy.$$

In particular for  $k = 1$ , by the *Hardy-type Inequality*, the following holds

$$\begin{aligned} \int \rho y^{4(\alpha-1)} (v')^2 dy &\leq C_3 \int \rho y^{2(\alpha-1)} (v'')^2 dy \\ &\leq C_3^2 \int \rho (v''')^2 dy. \end{aligned}$$

However, we want that

$$\int \rho y^2 (F')^2 dy \leq C_4 \int \rho (F''')^2 dy.$$

Hence we have

$$4(\alpha - 1) = 2,$$

which gives

$$\alpha = \frac{3}{2}.$$

For the general case, we have  $2k$  iterations and so

$$4k(\alpha - 1) = 2 \implies \alpha = \frac{2k+1}{2k}.$$

□

For the spectral results below, we always mean that the differential form for  $\mathbf{B}$  is equipped with the proper “radiation condition”, which shall be explained in Section 2.5.

**Lemma 2.3.2** (i) *The point spectrum of  $\mathbf{B}$  comprises of real eigenvalues only:*

$$\sigma_p(\mathbf{B}) = \{\lambda_\beta = -\frac{\beta}{2k+1}, \beta = 0, 1, 2, \dots\}. \quad (2.26)$$

*Eigenvalues are simple with eigenfunctions*

$$\psi_\beta(y) = \frac{(-1)^\beta}{\sqrt{\beta!}} D_y^\beta F(y).$$

(ii) *The set of eigenfunctions  $\Phi = \{\psi_\beta\}$  is complete in  $L_\rho^2(\mathbb{R})$ .*

(iii) *For any  $\lambda \notin \sigma(\mathbf{B})$ , the resolvent  $(\mathbf{B} - \lambda I)^{-1}$  is a compact operator in  $L_\rho^2(\mathbb{R})$ .*

**Proof.** (i) The existence of eigenvalues and eigenfunctions is found by applying

$D_y^\beta$  to (2.3)

$$D_y^\beta \mathbf{B}F \equiv \mathbf{B}D_y^\beta F + \frac{\beta}{2k+1} D_y^\beta F = 0.$$

It follows from the asymptotic expansion of (2.18) and (2.21), as  $\tau \rightarrow \infty$ , that no other eigenfunctions exist. Hence all eigenvalues are real and are given in (2.26).

(ii) We show that the system of eigenfunctions  $\{D^\beta F\}$  is complete in  $L_\rho^2(\mathbb{R})$ . By the Riesz-Fischer theorem we have to show that given a function  $G \in L_\rho^2(\mathbb{R})$ , then

$$\int D^\beta F(x)G(x) dx = 0 \quad \text{for any } \beta,$$

implies that  $G = 0$ . Let  $\mathcal{F}(\xi)$  and  $\mathcal{G}(\xi)$  be the Fourier transforms of  $F$  and  $G$ , then

$$\int \xi^\beta \mathcal{F}(\xi) \mathcal{G}(-\xi) d\xi = 0 \quad \text{for any } \beta.$$

Applying the Fourier transform to (2.3) we get

$$i \xi^{2k+1} \mathcal{F} + \frac{1}{2k+1} \xi D_\xi \mathcal{F} = 0.$$

Hence solving this, it is seen that  $\mathcal{F}(\xi) = e^{-i\xi^{2k+1}}$ . So,

$$\int \xi^\beta e^{-i\xi^{2k+1}} \mathcal{G}(-\xi) d\xi = 0 \quad \text{for any } \beta. \quad (2.27)$$

Then the function

$$M(z) = \int e^{-i\xi^{2k+1}} \mathcal{G}(-\xi) e^{i\xi z} dz,$$

is entirely analytic in  $\mathbb{C}$ , [42]. So (2.27) means that  $D^\beta M(0) = 0$  for any  $\beta$ , therefore  $M(z) \equiv 0$ . Hence  $\mathcal{G}(\xi) = 0$  almost everywhere and  $G = 0$ .

(iii) We do not go through it, but the proof follows that of the  $2m$ th-order case, set out in the paper [17]. Setting

$$m \mapsto k + \frac{1}{2},$$

will yield the same result. Note also that this also directly follows from the compact embedding of the corresponding spaces  $H_\rho^{2k+1} \subset L_\rho^2$ ; see Maz'ya's classic monograph on Sobolev spaces [49, p. 40].  $\square$



## 2.4 Spectrum and Polynomial Eigenfunctions of the Adjoint Operator $\mathbf{B}^*$

We now look to explicitly describe the eigenfunctions of the adjoint operator

$$\mathbf{B}^* = (-1)^{k+1} D_y^{2k+1} - \frac{1}{2k+1} y D_y. \quad (2.28)$$

### 2.4.1 Indefinite Metric

However, before we look at the operator  $\mathbf{B}^*$ , we first obtain the following easy observation:

**Proposition 2.4.1**  $\mathbf{B}^*$  is not adjoint to  $\mathbf{B}$  in the standard metric of  $L^2(\mathbb{R})$ .

**Proof.** Let  $v, w \in C_0^\infty(\mathbb{R})$ , then integration by parts yields

$$\begin{aligned} \langle \mathbf{B}v, w \rangle &\equiv \int_{\mathbb{R}} \left( (-1)^{k+1} v^{(2k+1)} + \frac{1}{2k+1} (yv)' \right) w \\ &= \int_{\mathbb{R}} \left( (-1)^k v^{(2k)} - \frac{1}{2k+1} yv \right) w' \\ &= \vdots \\ &= \int_{\mathbb{R}} v \left( (-1)^k w^{(2k+1)} - \frac{1}{2k+1} yw' \right) \\ &= \langle v, \tilde{\mathbf{B}}^* w \rangle, \end{aligned}$$

where

$$\tilde{\mathbf{B}}^* = (-1)^k D_y^{2k+1} - \frac{1}{2k+1} y D_y \neq \mathbf{B}^*.$$

□

In order to get the correct adjoint operator  $\mathbf{B}^*$ , it is necessary to use another metric. The scalar product of this *indefinite metric* is given by

$$\langle v, w \rangle_* = \int_{\mathbb{R}} v(y) \overline{w(-y)} dy. \quad (2.29)$$

Since  $\mathbf{B}$  and  $\mathbf{B}^*$  have real point spectrum, we may omit the complex conjugate here.

**Proposition 2.4.2**  $\mathbf{B}^*$  is adjoint to  $\mathbf{B}$  in the indefinite metric of  $\bar{L}^2(\mathbb{R})$ , with the scalar product (2.29).

**Proof.** For our operator, taking  $v, w \in C_0^\infty(\mathbb{R})$

$$\begin{aligned}
\langle \mathbf{B}v, w \rangle_* &= \int_{\mathbb{R}} \left( (-1)^{k+1} v^{(2k+1)}(y) + \frac{1}{2k+1} (yv(y))' \right) w(-y) \\
&= \int_{\mathbb{R}} \left( (-1)^{k+1} v^{(2k)}(y) + \frac{1}{2k+1} yv \right) w'(-y) \\
&= \vdots \\
&= \int_{\mathbb{R}} v(y) \left( (-1)^{k+1} w^{(2k+1)}(-y) - \frac{1}{2k+1} (-y)w'(-y) \right) \\
&= \langle v, \mathbf{B}^*w \rangle_*.
\end{aligned}$$

□

Thus,  $\mathbf{B}^*$  is adjoint to  $\mathbf{B}$  in the given indefinite metric, which we write down again as

$$\langle v, w \rangle_* = \int v(y) \overline{w(-y)} dy \equiv \langle v, \overline{Jw} \rangle \quad (v, w \in L_{\rho^*}^2). \quad (2.30)$$

Here, the *canonical symmetry operator*  $Jw(y) = w(-y)$  is bounded, self-adjoint and unitary (it is the *Gramm operator* of this metric). The condition  $Jw \in L_{\rho}^2$  determines the corresponding space  $L_{\rho^*}^2$  with the symmetric exponentially decaying weight

$$\rho^*(y) = e^{-a|y|^\alpha} \quad \text{for all } |y| \geq 1.$$

The set of even functions  $E_+ = \{v(-y) \equiv v(y)\}$  is a *positive lineal* (a linear manifold) of the metric (2.30),

$$\langle v, v \rangle_* > 0 \quad \text{for } v \in E_+, \quad v \neq 0,$$

and odd functions  $E_- = \{v(-y) \equiv -v(y)\}$  give the corresponding *negative lineal*. Therefore,  $L_{\rho^*}^2$  with this metric is *decomposable*,

$$v = v_+ + v_- \equiv \frac{v(y) + v(-y)}{2} + \frac{v(y) - v(-y)}{2}, \quad \text{where } v_{\pm} \in E_{\pm} \implies L_{\rho^*}^2 = E_+ \oplus E_-,$$

where, in addition,  $E_+ \perp E_-$  in the metric (2.30). The corresponding positive *majorizing* metric is given by

$$|\langle v, v \rangle_*| \leq [v, v]_* \equiv \langle v_+, v_+ \rangle_* - \langle v_-, v_- \rangle_*,$$

etc. This case of the decomposable space with indefinite metric with straightforward majorizing one is treated as rather trivial; see Azizov-Iokhvidov [3] for linear operators theory in spaces with indefinite metric. Metric (2.30) is widely used therein, [3, p. 13, 17, 23, 114]. Then the domain of  $B^*$  is defined as  $H_{\rho^*}^3$ , etc.

**Historical Remark:** As we mentioned, basic results of linear operator theory in spaces with indefinite metrics can be found in Azizov and Iokhvidov. It wasn't until about 1944 that L.S. Pontryagin published the article on “Hermitian operators in spaces with indefinite metric” [52]. A new area of operator theory had been formed from Pontryagin's studies. This work set by Pontryagin was continued in the 1950s by M.G. Krein [46] and I.S. Iokhvidov [34].

## 2.4.2 Discrete Spectrum and Polynomial Eigenfunctions of $B^*$

We consider the spectrum of the linear adjoint operator  $B^*$  in the weighted space  $L_{\rho^*}^2(\mathbb{R})$ , with exponentially decaying weight, which has been already introduced above,

$$\rho^*(y) = e^{-a|y|^\alpha} \quad \text{for } |y| \geq 1.$$

Here we have that  $\rho^*(y) > 0$  and  $a \in (0, 2d)$ , is a sufficiently small positive constant.

**Lemma 2.4.3**  $B^*$  is a bounded linear operator from  $H_{\rho^*}^{2k+1}(\mathbb{R})$  to  $L_{\rho^*}^2(\mathbb{R})$ .

**Proof.** The proof is the same as that for the operator  $B$  in Lemma 2.3.1. We look at the adjoint operator

$$B^*v = (-1)^{k+1}v^{(2k+1)} - \frac{1}{2k+1}yv'.$$

For  $B^*$  to be bounded we require

$$\int \rho^*(yv')^2 \leq \tilde{C} \int \rho^*(v^{(2k+1)})^2 dy,$$

where  $\tilde{C}$  is a constant. To show this, we look at

$$\begin{aligned} 0 &\leq \int \rho^*(v' + y^{\tilde{\gamma}}v)^2 dy \\ &= \int \rho^*((v')^2 + y^{2\tilde{\gamma}}v^2 + 2y^{\tilde{\gamma}}v'v) dy, \end{aligned}$$

for some positive exponential  $\tilde{\gamma}$ . Integrating by parts in the last term, we have

$$\begin{aligned} 2 \int (\rho^* y^{\tilde{\gamma}}) v' v dy &= \int (\rho^* y^{\tilde{\gamma}}) (v^2)' dy \\ &= - \int v^2 (\rho^* y^{\tilde{\gamma}})' dy. \end{aligned}$$

For exponential weight

$$\rho^*(y) = e^{-ay^\alpha},$$

we have that

$$(\rho^* y^{\tilde{\gamma}})' \sim e^{-ay^\alpha} y^{\tilde{\gamma}+\alpha-1},$$

for  $y \gg 1$  and similarly for  $y \ll -1$ , with  $y \mapsto |y|$ . Therefore

$$\int \rho^*(v')^2 dy + \int \rho^* y^{2\tilde{\gamma}} v^2 dy + \tilde{C}_1 \int \rho^* y^{\tilde{\gamma}+\alpha-1} v^2 dy \geq 0.$$

Equating powers of  $y$ , we find

$$\tilde{\gamma} = \alpha - 1.$$

Substituting for  $\tilde{\gamma}$  we have that

$$\int \rho^* y^{2(\alpha-1)} v^2 dy \leq \tilde{C}_2 \int \rho^*(v')^2 dy.$$

Looking at the particular case for  $k = 1$ , we have that

$$\begin{aligned} \int \rho^* y^{4(\alpha-1)} (v')^2 dy &\leq \tilde{C}_3 \int \rho^* y^{2(\alpha-1)} (v'')^2 dy \\ &\leq \tilde{C}_3^2 \int \rho^*(v''')^2 dy. \end{aligned}$$

We continue to obtain

$$\int \rho^* y^2 (v')^2 dy \leq \tilde{C}_4 \int \rho^*(v''')^2 dy,$$

hence once again we have that

$$4(\alpha - 1) = 2,$$

which gives

$$\alpha = \frac{3}{2}.$$

For the general case, we have  $2k$  iterations and so

$$4k(\alpha - 1) = 2 \implies \alpha = \frac{2k+1}{2k}.$$

□

As stated before in (2.2.1), we can find an easier method to determine the polynomial adjoint eigenfunctions  $\{\psi_\beta^*\}$  (generalised Hermite polynomials). In order to derive these adjoint eigenfunctions, we apply the Fourier transform to the eigenvalue problem

$$\mathbf{B}^* u = \lambda u, \tag{2.31}$$

so that we obtain a first-order equation of the form

$$\frac{1}{2k+1} \xi V' + \left( \frac{1}{2k+1} - i\xi^{2k+1} \right) V = \lambda V. \tag{2.32}$$

The general solution of this is given by

$$V(\xi) = A|\xi|^{(2k+1)\lambda-1} e^{i\xi^{2k}\xi},$$

where  $A = A\left(\frac{\xi}{|\xi|}\right)$  is an arbitrary smooth function. We observe that  $V(\xi)$  is a sufficiently good function at both singular points  $\xi = 0$  and  $\xi = \pm\infty$ , for  $A = 0$  only. Hence the only distributions that satisfy (2.32) correspond to  $A = 0$ , i.e., the only distributional solution must have point support.

Therefore, by distribution theory, any solution  $u(y)$  must be a polynomial. If its degree is  $\beta$ , then

$$u(y) = \sum_{j=0}^s P_j(y),$$

where  $P_j(y)$  is a homogeneous polynomial and  $s = \lfloor \frac{\beta}{2k+1} \rfloor$ . From the eigenvalue problem 2.31, we can work out all terms of the polynomial, for a given degree  $l$ .

Since

$$(-1)^{k+1} D_y^{2k+1} u - \frac{1}{2k+1} y D_y u = \lambda_\beta u.$$

Then

$$\lambda_\beta = -\frac{\beta}{2k+1}, \quad k = 0, 1, 2, \dots, \quad (2.33)$$

and we can define all other polynomials  $P_j(y)$  by

$$P_j(y) = \frac{(-1)^{(k+1)}}{j!} D^{(2k+1)j} P_0(y), \quad j = 1, \dots, s.$$

Fixing  $P_0(y) = y^\beta$ , we find that for the eigenfunctions  $\{\psi_\beta\}$  of  $\mathbf{B}^*$ , the corresponding adjoint eigenfunctions  $\{\psi_\beta^*\}$  are given by

$$\psi_\beta^*(y) = \frac{1}{\sqrt{\beta!}} \left[ y^\beta + (-1)^{(k+1)} \sum_{j=1}^{\lfloor \frac{\beta}{2k+1} \rfloor} \frac{1}{j!} D^{(2k+1)j} y^\beta \right]. \quad (2.34)$$

For convenience, we state the following result:

**Proposition 2.4.4**  $\mathbf{B}^*$  has the point spectrum  $\sigma_p(\mathbf{B}^*)$ , given by (2.33) with polynomial eigenfunctions (2.34) of order  $\beta$ .

To get such a discrete spectrum, i.e., to prove that

$$\sigma(\mathbf{B}^*) = \sigma_p(\mathbf{B}^*),$$

a “radiation condition”, to be specified in Section 2.5, is necessary.

### 2.4.3 On Bi-Orthonormality Property by Extension of Linear Functionals

As usual in linear operator theory (the non self-adjoint case) [31], having obtained complete and closed sets of eigenfunctions of the operator pair  $\{\mathbf{B}, \mathbf{B}^*\}$ , the next natural step includes defining their scalar products in the corresponding indefinite metric  $\langle \cdot, \cdot \rangle_*$ . This means to look first at the products

$$\langle w_0, \psi_\gamma^* \rangle_*,$$

as values of the linear functional  $\psi_\gamma^*$  at the elements  $w_0 \in L_{\rho^*}^2$ . As we have seen in the eigenfunction expansion sections, these products are well defined in the

standard integral sense. As the second step, we perform a standard procedure of extension of such uniformly convex functionals by the classic Hahn-Banach theorem in linear normed space<sup>1</sup>, [42, 47]. In view of the density of  $L^2_\rho$ , such an extension of the linear functional is then expected to be uniquely defined.

In general, we expect that it is possible to define the standard bi-orthonormality relation between bases:

$$\langle \psi_\beta, \psi_\gamma^* \rangle_* = \delta_{\beta\gamma} \quad \text{for any } \beta \text{ and } \gamma, \quad (2.35)$$

where  $\delta_{\beta\gamma}$  is the Kronecker delta. This also defines such functionals  $\langle v, \psi_\gamma^* \rangle_*$  for any  $v \in L^2_\rho$ . The actual meaning of the integrals in (2.35) (in the v.p. sense, a non-standard *principal value* sense) is not obvious and can be tricky. In what follows, we do not pay any essential attention to such extensions of linear functional properties, since we are not going to rely on these later on.

Note that explaining (2.35) is not that far away from typical ideas of regularisation of oscillatory integrals in classic theory of pseudo-differential operators; see e.g., Shubin [63]. According to these methods, the orthogonality in (2.35), is understood according to standard regularisation of divergent integrals in distribution theory. Nevertheless, a full justification of such properties can be difficult and falls out of the framework of the present research.

However, before we look at this bi-orthonormality condition of dual-space, we first note the following remark:

Due to the indefinite metric  $\langle \cdot, \cdot \rangle_*$ , we must slightly revise the definition of the eigenfunctions  $\psi_\beta(y)$ . Looking back once again at the expansion of the convolution, where the rescaled solution is given by (2.18), we note the definition of the moments of the initial data, which give rise to the adjoint eigenfunctions,  $\psi_\beta^*(y)$ . These moments (2.19), are given by

$$M_\beta(u_0) \equiv \langle u_0(z), \psi_\beta^*(z) \rangle = \frac{1}{\sqrt{\beta!}} \int_{\mathbb{R}} z^\beta u_0(z) \, dz.$$

---

<sup>1</sup>“If  $X$  is a linear normed space,  $L$  is a linear manifold and  $f$  is a linear continuous functional defined on  $L$ , then  $f$  can be extended to  $F$  on  $X$  and  $\|F\|_X = \|f\|_L$ ”.

However we must have the representation in the indefinite metric, such that

$$\begin{aligned} M_\beta(u_0) &\equiv \langle u_0(z), \psi_\beta^*(z) \rangle_* = \frac{1}{\sqrt{\beta!}} \int_{\mathbb{R}} (-z)^\beta u_0(z) \, dz \\ &= \frac{(-1)^\beta}{\sqrt{\beta!}} \int_{\mathbb{R}} z^\beta u_0(z) \, dz. \end{aligned}$$

Therefore in the expansion, there must an extra term of  $(-1)^\beta$ , for the dual products. So for purposes of convenience we now have to take the eigenfunctions in the following form (i.e., the multiplier  $(-1)^\beta$  is omitted):

$$\psi_\beta(y) = \frac{1}{\sqrt{\beta!}} D_y^\beta F(y), \quad \beta \geq 0. \quad (2.36)$$

This replaces our original definition (2.15), without loss of previous results.

**Proposition 2.4.5**

$$\langle \psi_\beta, \psi_\beta^* \rangle_* = 1 \quad \text{for all } \beta \geq 0.$$

**Proof.** We start by looking at the scalar product, in the indefinite metric, defined by

$$\langle \psi_\beta, \psi_\beta^* \rangle_* = \int_{\mathbb{R}} \psi_\beta(y) \psi_\beta^*(-y) \, dy.$$

By our definitions of  $\psi_\beta(y)$  (2.36) and  $\psi_\beta^*(y)$  (2.34), we substitute to find that

$$\langle \psi_\beta, \psi_\beta^* \rangle_* = \int \frac{1}{\sqrt{\beta!}} D_y^\beta F(y) \frac{1}{\sqrt{\beta!}} \left[ (-y)^\beta + (-1)^{(k+1)} \sum_{j=1}^{\lfloor \frac{\beta}{2k+1} \rfloor} \frac{1}{j!} D^{(2k+1)j} (-y)^\beta \right] dy,$$

Let us now look at the product, where we apply to  $\psi_\beta^*$ , the identity operator

$$(D_y^\beta)^{-1} D_y^\beta = I,$$

with a standard definition and construction of the inverse integral operator  $(D_y^\beta)^{-1} = D_y^{-\beta}$ , such that

$$\langle \psi_\beta, \psi_\beta^* \rangle_* = \langle \psi_\beta, D_y^{-\beta} D_y^\beta \psi_\beta^* \rangle_*.$$



Integrating by parts,  $\beta$  times, we find that

$$\begin{aligned}
\langle \psi_\beta, \psi_\beta^* \rangle_* &= \int \left[ \frac{1}{\sqrt{\beta!}} D_y^\beta F(y) D_y^{-\beta} D_y^\beta \frac{1}{\sqrt{\beta!}} ((-y)^\beta + \dots) \right] dy \\
&= \frac{1}{\beta!} \int (-1) \left[ D_y^{\beta-1} F(y) D_y^{-\beta+1} D_y^\beta ((-y)^\beta + \dots) \right] dy \\
&\vdots \\
&= \frac{1}{\beta!} \int (-1)^\beta \left[ F(y) D_y^\beta ((-y)^\beta + \dots) \right] dy \\
&= \frac{1}{\beta!} \int (-1)^\beta F(y) (-1)^\beta \beta! dy.
\end{aligned}$$

One can see that, according to these formal calculus of integration by parts, we each time improve the convergence properties of the integrals involved, meaning using a distributional treatment of those integrals as values of certain linear functionals, as generalised functions (distributions). As customary, this corresponds to a regularisation of divergent integrals.

Hence, eventually, it follows that

$$\langle \psi_\beta, \psi_\beta^* \rangle_* = \int F(y) dy = 1,$$

so finally we arrive at a convergent (but not absolutely, i.e.,  $F$  is not Lebesgue measurable in  $\mathbb{R}$ , as we have seen) integral.

□

**Proposition 2.4.6** *In terms of the above formal calculus,*

$$\langle \psi_\beta, \psi_\gamma^* \rangle_* = 0 \quad \text{for all } \beta \neq \gamma.$$

**Proof.** The first part of the proof follows that of Proposition 2.4.5.

First consider the case where  $\beta > \gamma$ . After integration by parts, it can be seen that  $\langle \psi_\beta, \psi_\gamma^* \rangle_*$  may be written as

$$\langle \psi_\beta, \psi_\gamma^* \rangle_* = (-1)^\beta \int \frac{(-1)^\beta}{\sqrt{\beta!}} D_y^{-\beta} D_y^\beta F(y) \frac{1}{\sqrt{\gamma!}} D_y^\beta ((-y)^\gamma + \dots) dy.$$

However since  $\beta > \gamma$ , then it is known that

$$D_y^\beta (-y)^\gamma = 0.$$

So it follows that (2.4.6) holds true.

Now consider the case when  $\beta < \gamma$ . In this case it can easily be seen that the above argument will not work. Rather than attempting to use a similar argument, we instead use another proof which encompasses both cases of  $\beta > \gamma$  and  $\beta < \gamma$ .

By the definitions of the linear operator  $\mathbf{B}$  and the adjoint operator  $\mathbf{B}^*$ , we know that

$$\begin{cases} \mathbf{B}\psi_\beta = \lambda_\beta\psi_\beta, \\ \mathbf{B}^*\psi_\gamma^* = \lambda_\gamma\psi_\gamma^*, \end{cases} \quad (2.37)$$

which defines the eigenvalue problems for these two operators. It can easily be seen that by taking the scalar product of (2.37), in the indefinite metric, with  $\psi_\gamma^*$  and  $\psi_\beta$ , respectively, yields

$$\begin{cases} \langle \mathbf{B}\psi_\beta, \psi_\gamma^* \rangle_* = \lambda_\beta \langle \psi_\beta, \psi_\gamma^* \rangle_*, \\ \langle \psi_\beta, \mathbf{B}^*\psi_\gamma^* \rangle_* = \lambda_\gamma \langle \psi_\beta, \psi_\gamma^* \rangle_*. \end{cases}$$

However from the definition of the adjoint operator  $\mathbf{B}^*$ , we know that

$$\langle \mathbf{B}\psi_\beta, \psi_\gamma^* \rangle_* = \langle \psi_\beta, \mathbf{B}^*\psi_\gamma^* \rangle_*.$$

Hence, if  $\beta \neq \gamma$  (i.e.,  $\lambda_\beta \neq \lambda_\gamma$ ), it must follow that

$$\langle \psi_\beta, \psi_\gamma^* \rangle_* = 0 \quad \text{for all } \beta \neq \gamma.$$

Again, when necessary, we assume a proper regularisation of the integrals, which are treated as values of some linear functionals.

□

**Corollary 2.4.7** *The orthonormality condition*

$$\langle \psi_\beta, \psi_\gamma^* \rangle_* = \delta_{\beta\gamma},$$

*holds true, in the indefinite metric, for all  $\beta, \gamma \geq 0$ .*

This follows directly from Propositions 2.4.5 and 2.4.6.

## 2.5 Generalised “Radiation Conditions” for the BVP problem

As we have promised, we now clarify the “radiation-type conditions” posed at infinity, which allow the operator pair  $\{\mathbf{B}, \mathbf{B}^*\}$  to have purely discrete spectra  $\{-\frac{l}{2k+1}, l \geq 0\}$ , already detected by eigenfunction expansion of the corresponding semigroups.

### 2.5.1 Linear Operator $\mathbf{B}$

To this end, in order for our linear ODE to be “well-posed” (i.e., with a proper number of boundary conditions at infinity), both in a mathematical sense, as well as a physical sense, we look for conditions, which must be satisfied for the eigenvalue equation:

$$\mathbf{B}\psi_l(y) = \lambda_l \psi_l(y) \quad \text{in } \mathbb{R}.$$

This can be rewritten as

$$(-1)^{k+1} \psi_l^{(2k+1)} + \frac{1}{2k+1} \psi_l + \frac{1}{2k+1} y \psi_l' = \lambda_l \psi_l. \quad (2.38)$$

Since the order of this ODE is  $2k+1$ , it is natural that there must also be  $2k+1$  boundary conditions placed, as classic theory of ordinary differential operators suggests; see Naimark’s monograph [50].

ELIMINATING EXPONENTIALLY GROWING BUNDLES. First consider the problem, as  $y \rightarrow +\infty$ . Attempting to balance leading order terms in (2.38), leads to

$$(-1)^{k+1} \psi_l^{(2k+1)} + \frac{1}{2k+1} y \psi_l' \sim 0. \quad (2.39)$$

As  $y \rightarrow +\infty$ , we have that

$$\psi_l(y) \sim e^{by^{\frac{2k+1}{2k}}}, \quad b \in \mathbb{C}, \quad b \neq 0, \quad (2.40)$$

hence substituting this into the above equation, yields

$$\begin{aligned} (-1)^{k+1} \left( \frac{2k+1}{2k} b \right)^{2k+1} + \frac{1}{2k+1} \left( \frac{2k+1}{2k} b \right) &\sim 0 \\ \implies b^{2k} &\sim (-1)^k \left( \frac{2k}{2k+1} \right)^{2k} \frac{1}{2k+1}. \end{aligned}$$

It can be seen that we end up with two cases, dependent on the value of the

parameter  $k$ , which will determine the sign of  $b^{2k}$  and therefore its roots. Hence, for now, we ignore the term  $\left(\frac{2k}{2k+1}\right)^{2k} \frac{1}{2k+1}$  and just look at the value of  $(-1)^k$ , assuming that

$$b = \left(\frac{2k}{2k+1}\right)^{2k} \frac{1}{2k+1} \hat{b}.$$

*When  $k$  is even:* for even values of  $k$ , it is noted that

$$\hat{b}^{2k} = 1.$$

Hence there must  $2k$  roots for  $\hat{b}$ , which are given by

$$\hat{b}_m = e^{\frac{\pi m i}{k}},$$

for  $m = 0, 1, \dots, 2k - 1$ .

*When  $k$  is odd:* for odd values of  $k$ , there is now a negative sign, such that

$$\hat{b}^{2k} = -1.$$

Similarly, as before, we derive  $2k$  roots, where now

$$\hat{b}_m = e^{\frac{(\pi+2\pi m)i}{2k}}, \quad \text{for } m = 0, 1, \dots, 2k - 1.$$

For the problem to be well posed on the space  $L_\rho^2$ , it is important to look for roots such that there is exponential decay, rather than growth. In other words, it must satisfy the condition  $\rho = e^{-ay^\alpha}$ , as  $y \rightarrow +\infty$ . Hence we need to eliminate any roots such that  $\text{Re } \hat{b}_m > 0$ . It is first noted that equality,  $\text{Re } \hat{b}_m = 0$ , occurs when

$$\frac{\pi}{2} = \begin{cases} \frac{\pi m}{k} & \text{for even } k, \\ \frac{(\pi+2\pi m)}{2k} & \text{for odd } k, \end{cases}$$

with the same applying for  $\frac{3\pi}{2}$ . For  $\text{Re } \hat{b}_m = 0$ , we must have that

$$m = \begin{cases} \frac{k}{2} & \text{for even } k, \\ \frac{k-1}{2} & \text{for odd } k \end{cases}$$

and

$$m = \begin{cases} \frac{3k}{2} & \text{for even } k, \\ \frac{3k-1}{2} & \text{for odd } k. \end{cases}$$

Hence in order to eliminate roots which give rise to exponential growth, the conditions placed must be such that we do not include roots such that

$$\begin{cases} m < \frac{k}{2} & \text{and} & m > \frac{3k}{2} & \text{for even } k, \\ m < \frac{k-1}{2} & \text{and} & m > \frac{3k-1}{2} & \text{for odd } k. \end{cases}$$

Therefore, by taking the weight

$$\rho(y) = e^{-ay^{\frac{2k+1}{2k}}}, \quad \text{with any sufficiently small } a > 0 \quad (2.41)$$

we eliminate all exponentially growing oscillatory bundles. A precise bound on admissible  $a > 0$  will be derived below. Thus, we have  $k - 1$  conditions placed here, satisfying (2.39) as  $y \rightarrow +\infty$ .

Similarly, we can do the same analysis for  $y \rightarrow -\infty$ . As  $y \rightarrow -\infty$ , we have that

$$\psi(y) \sim e^{b(-y)^{\frac{2k+1}{2k}}} \quad (2.42)$$

and substituting into equation (2.39), yields

$$b^{2k} \sim (-1)^{k+1} \left( \frac{2k}{2k+1} \right)^{2k} \frac{1}{2k+1}.$$

It can be seen (as expected), that this only differs from the  $y \rightarrow +\infty$  case by a differing sign. This leads to

$$\hat{b}_m = \begin{cases} e^{\frac{(\pi+2\pi m)i}{2k}}, & \text{for even } k, \\ e^{\frac{\pi mi}{k}}, & \text{for odd } k, \end{cases}$$

for  $m = 0, 1, \dots, 2k - 1$ .

As before we do not want roots such that  $\text{Re } \hat{b}_m > 0$ , where in this case

$$\begin{cases} m < \frac{k-1}{2} & \text{and} & m > \frac{3k-1}{2} & \text{for even } k, \\ m < \frac{k}{2} & \text{and} & m > \frac{3k}{2} & \text{for odd } k. \end{cases}$$

Hence for the weight

$$\rho(y) = e^{a|y|^{\frac{2k+1}{2k}}}, \quad \text{with sufficiently small } a > 0, \quad (2.43)$$

we eliminate these conditions. This leads to a further  $k$  conditions, which are placed at  $y \rightarrow -\infty$ . All these correspond to eliminating exponentially growing asymptotic bundles, though that is not enough as we explain below.

**RADIATION CONDITIONS.** As stated before, we look for  $2k + 1$  conditions to be posed onto the problem. Hence there are two more conditions needed. These conditions at  $y \rightarrow \pm\infty$  are known as *radiation conditions*. In classic problems of quantum mechanics, acoustics, and physics, the general idea behind radiation conditions is that energy sources must exactly be that and not sinks of energy. Hence all energy must be radiated from a point and scatter to infinity. We refer to the book by Sommerfeld who (in 1912) first proposed radiation conditions for the *Helmholtz equation*, [64]. We also refer to the paper by Xing, which applies the radiation condition [68] and where more references may be found.

In our problem, the radiation conditions are rather tricky and have almost nothing to do with the classic ones. We recall that we identify those just for convenience (to verify the domain of  $\mathbf{B}$  and  $\mathbf{B}^*$ ), since the eigenfunction expansions of the semigroups, as the main tool of our asymptotic analysis, automatically includes the necessary two conditions at infinity, as we show below.

The origin of the radiation condition for  $\mathbf{B}$  is as follows: Let us now balance all lower-order terms in the eigenvalue problem (2.38), so that

$$\frac{1}{2k+1} \psi_l + \frac{1}{2k+1} y \psi'_l \sim \lambda_l \psi_l.$$

By integration we can easily see that

$$\psi_l(y) \sim A y^{(2k+1)\lambda_l - 1}, \quad (2.44)$$

for some constant  $A$ . Of course, this corresponds to the obvious root  $b = 0$  in the exponential expansions (2.40) and (2.42).

We note that (2.44), is a “rational” function, unlike the exponentially oscillatory bundles in (2.40) and (2.42). For  $y \rightarrow -\infty$ , we know that, for any  $\lambda_l \in \mathbb{C}$ , rational solutions such as (2.44) do not belong to the space  $L^2_\rho$  with the exponentially growing weight (2.43).

Thus, overall, we conclude as follows:

$$\text{at } y = -\infty, \text{ the proper weight (2.43) generates } k + 1 \text{ conditions.} \quad (2.45)$$

So, this is a usual and a standard situation, so that the singular point  $y = -\infty$  does not require any radiation-type condition. This is not the case for the “oscillatory” end-point  $y = +\infty$ .

For  $y \rightarrow +\infty$ , consider all complex “eigenvalues”,  $\lambda \in \mathbb{C}$ , such that  $\lambda = P + iQ$ , for some  $P, Q \in \mathbb{R}$ . From (2.44),  $\psi_l(y)$  may now be given by

$$\psi_l(y) \sim y^{\hat{P} + i\hat{Q}}, \quad (2.46)$$

for  $\hat{P}, \hat{Q} \in \mathbb{R}$ . Hence we see that

$$\begin{aligned} \psi_l(y) &\sim y^{\hat{P}} e^{i\hat{Q} \ln y} \\ &\sim y^{\hat{P}} [\cos(\hat{Q} \ln y) + i \sin(\hat{Q} \ln y)], \end{aligned} \quad (2.47)$$

as  $y \rightarrow +\infty$ . However we know that, from the asymptotic analysis, the behaviour of proper eigenfunctions is different and given by a different type of highly oscillatory functions:

$$\psi_l(y) \sim y^{-\frac{2k-1}{4k}} \cos(y^{\frac{2k+1}{2k}}),$$

which obviously gives a stronger oscillatory behaviour than a pure  $\cos(\hat{Q} \ln y)$  in (2.47). However, (2.47) admits weaker oscillatory behaviour and so we must place a condition to eliminate this behaviour. We recall all the proper eigenfunctions being given by the generating formula  $\psi_l(y) = \frac{1}{\sqrt{l!}} D^l F(y)$ , *do not contain the bundle* (2.44), since the fundamental rescaled kernel  $F(y)$  does not by the known divergence of the operator  $\mathbf{B}$  (the equation for  $F$  has been integrated once with the zero constant of integration that eliminated any trace of (2.44)); see computations below.

Thus, the *generalised radiation condition*, that is necessary for discreteness of the spectrum of  $\mathbf{B}$  in  $L_\rho^2$ , can be formulated as follows:

*For the eigenvalue equation (2.38), the bundle (2.46) must be absent at  $y = +\infty$ .*

Actually, it is easy to see that all our eigenfunctions  $\{\psi_l(y)\}$  satisfy this condition. Indeed,  $F(y) \equiv \psi_0(y)$  does satisfy this, by integrating once, where we have that

$$\begin{aligned} (-1)^{k+1} F^{(2k+1)} + \frac{1}{2k+1} (Fy)' &= 0 \\ \implies (-1)^{k+1} F^{(2k)} + \frac{1}{2k+1} Fy &= C, \end{aligned}$$

for some constant of integration  $C$ . The last term precisely shows that such a rational behaviour is absent, since we have that

$$F(y) \sim (2k+1) \frac{C}{y},$$

which implies that  $C \equiv 0$ . Then each eigenfunction

$$\psi_l(y) = \frac{1}{\sqrt{l!}} D_y^l F(y) \quad \text{for all } l \geq 0,$$

also satisfies this condition.

### 2.5.2 Conditions on the Adjoint Operator, $\mathbf{B}^*$

We now more briefly apply the same analysis to the adjoint operator  $\mathbf{B}^*$ , where the eigenvalue problem is now given by

$$(-1)^{k+1} \psi_l^{*(2k+1)} - \frac{1}{2k+1} y \psi^{*'} = \lambda_l \psi^*. \quad (2.48)$$

Similarly, as before, we look at exponential bundles with  $b \neq 0$ :

$$\psi^*(y) \sim e^{by^{\frac{2k+1}{2k}}},$$

as  $y \rightarrow +\infty$  and

$$\psi^*(y) \sim e^{b|y|^{\frac{2k+1}{2k}}},$$

as  $y \rightarrow -\infty$ .

Balancing terms

$$(-1)^{k+1} \psi_l^{*(2k+1)} \sim \frac{1}{2k+1} y \psi^{*'}, \quad (2.49)$$

it can easily be seen that the case for the adjoint operator  $\mathbf{B}^*$ , only differs from the linear operator  $\mathbf{B}$ , with respect to a change of sign. Hence it is seen that, as  $y \rightarrow +\infty$ ,

$$\hat{b}_m = \begin{cases} e^{\frac{(\pi+2\pi m)i}{2k}}, & \text{for even } k, \\ e^{\frac{\pi m i}{k}}, & \text{for odd } k, \end{cases}$$



for  $m = 0, 1, \dots, 2k - 1$ . When  $y \rightarrow -\infty$ , we have that

$$\hat{b}_m = \begin{cases} e^{\frac{\pi m i}{k}}, & \text{for even } k, \\ e^{\frac{(\pi + 2\pi m)i}{2k}}, & \text{for odd } k, \end{cases}$$

for  $m = 0, 1, \dots, 2k - 1$ .

It is noted that we look at the problem in the space  $L_{\rho^*}^2$ , with weight

$$\rho^*(y) = e^{-a|y|^{\frac{2k+1}{2k}}}, \quad a > 0 \text{ is small enough.}$$

Hence for this weight, for  $y \rightarrow +\infty$ , we eliminate the roots such that

$$\begin{cases} m < \frac{k-1}{2} & \text{and} & m > \frac{3k-1}{2} & \text{for even } k, \\ m < \frac{k}{2} & \text{and} & m > \frac{3k}{2} & \text{for odd } k \end{cases}$$

and for  $y \rightarrow -\infty$

$$\begin{cases} m < \frac{k}{2} & \text{and} & m > \frac{3k}{2} & \text{for even } k, \\ m < \frac{k-1}{2} & \text{and} & m > \frac{3k-1}{2} & \text{for odd } k. \end{cases}$$

These give  $2k - 1$  conditions in total.

Whilst before, balancing the rest of the terms will lead to our radiation conditions, this is not possible to do in similar lines (as for  $\mathbf{B}$ ) for the case of  $\mathbf{B}^*$ . It is noted that for

$$-\frac{1}{2k+1} y \psi^{*'}(y) \sim \lambda_l \psi^*(y),$$

after integration, it can be shown that

$$\psi_l^*(y) \sim A_0 y^{-(2k+1)\lambda_l},$$

for some constant  $A_0$ . However, this behaviour is perfectly acceptable in  $L_{\rho^*}^2$ , and it is satisfied by the polynomial adjoint eigenfunctions  $\{\psi_l^*(y)\}$ , which are our generalised Hermite polynomials.

Instead we look for the remaining two conditions for the problem as  $y \rightarrow -\infty$ . In light of the definition of  $\mathbf{B}^*$ , we assume to have another type of radiation condition, which for convenience we denote as the “*Generalised Adjoint Radiation*”

Condition”, such that we

exclude two bundles with “maximum” oscillatory components at  $y = -\infty$ .

Again, this extra radiation condition makes the total number of conditions to be equal to  $2k + 1$ , which is the differential order of the operator  $\mathbf{B}^*$ , so that the eigenvalue problem becomes algebraically consistent [50]. In other words, we have an algebraic inhomogeneous system of  $2k + 1$  equations with analytic coefficients with  $2k + 1$  unknowns. Such systems do not have more than a countable set of solutions, which are eigenvalues of  $\mathbf{B}^*$ , which is defined in such a way.

Hence, we now restrict those roots such that  $\text{Re } \hat{b}_m = 0$ . Hence the conditions, as  $y \rightarrow +\infty$  are now given by the following distribution of the acceptable coefficients  $\{b_m\}$ :

$$\begin{cases} m \leq \frac{k}{2} & \text{and} & m \geq \frac{3k}{2} & \text{for even } k, \\ m \leq \frac{k-1}{2} & \text{and} & m \geq \frac{3k-1}{2} & \text{for odd } k. \end{cases}$$

### 2.5.3 Calculations for the Weights, $\rho(y)$ and $\rho^*(y)$

We now determine the sharp distance between the principle root  $b_{m_c}$  such that  $\text{Re } b_{m_c} = 0$  and the previous root  $b_{m_c-1}$ , where  $\text{Re } b_{m_c-1} > 0$ . In doing so we may find our weight  $\rho(y)$  such that it cuts off all unwanted roots, for which  $\text{Re } b > 0$ .

First consider the case as  $y \rightarrow +\infty$ . For  $\text{Re } b_{m_c} = 0$ , it is known that there is a root here and this is given by

$$b_{m_c} = d_k i,$$

where  $m_c$  is given to be

$$m_c = \begin{cases} \frac{k}{2} & \text{for even } k, \\ \frac{k-1}{2} & \text{for odd } k. \end{cases}$$

Here

$$d_k = 2k \left( \frac{1}{2k+1} \right)^{\frac{2k+1}{2k}},$$

as before. Hence it can easily be seen that for the root  $m_c - 1$ ,

$$m_c - 1 = \begin{cases} \frac{k-2}{2} & \text{for even } k, \\ \frac{k-3}{2} & \text{for odd } k, \end{cases}$$

or

$$m_c - 1 = \lfloor \frac{k-2}{2} \rfloor,$$

for all  $k$ . This yields

$$\hat{b}_{m_c-1} = \begin{cases} d_k \left[ \cos\left(\frac{k-2}{2} \frac{\pi}{k}\right) + i \sin\left(\frac{k-2}{2} \frac{\pi}{k}\right) \right] & \text{for even } k, \\ d_k \left[ \cos\left(\lfloor \frac{k-2}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}\right) + i \sin\left(\lfloor \frac{k-2}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}\right) \right] & \text{for odd } k. \end{cases}$$

Hence the distance between the the real axis and these roots is

$$d = \begin{cases} d_k \cos\left(\frac{k-2}{2} \frac{\pi}{k}\right) & \text{for even } k, \\ d_k \cos\left(\lfloor \frac{k-2}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}\right) & \text{for odd } k, \end{cases}$$

as  $y \rightarrow +\infty$ .

For  $y \rightarrow -\infty$ , we do not have any roots  $\text{Re } b_{m_c} = 0$ , for any  $k$ . However we look at the distance between the real axis and the next root such that  $\text{Re } b > 0$ .

In this case we find the distance between the real axis and roots such that

$$m_c - 1 = \lfloor \frac{k-1}{2} \rfloor,$$

for all  $k$ . Hence we have that

$$\hat{b}_{m_c-1} = \begin{cases} d_k \left[ \cos\left(\lfloor \frac{k-1}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}\right) + i \sin\left(\lfloor \frac{k-1}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}\right) \right] & \text{for even } k, \\ d_k \left[ \cos\left(\frac{k-1}{2} \frac{\pi}{k}\right) + i \sin\left(\frac{k-1}{2} \frac{\pi}{k}\right) \right] & \text{for odd } k. \end{cases}$$

This gives the distance between this root and the real axis

$$d = \begin{cases} d_k \cos\left(\lfloor \frac{k-1}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}\right) & \text{for even } k, \\ d_k \cos\left(\frac{k-1}{2} \frac{\pi}{k}\right) & \text{for odd } k. \end{cases}$$

This characterises our weighted space  $L_\rho^2(\mathbb{R})$ , with the exponential weight

$$\rho(y) = \begin{cases} e^{a|y|^{\frac{2k+1}{2k}}} & \text{for } y \leq -1, \\ e^{-ay^{\frac{2k+1}{2k}}} & \text{for } y \geq 1, \end{cases}$$

where  $a \in (0, 2d)$ .

Similarly we can find the weight  $\rho^*(y)$ . We note that the calculations are exactly the same as in the case for  $\rho(y)$ , except a difference in sign when calculating the roots. This leads to

$$d = \begin{cases} d_k \cos(\lfloor \frac{k-1}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}) & \text{for even } k, \\ d_k \cos(\frac{k-1}{2} \frac{\pi}{k}) & \text{for odd } k \end{cases}$$

as  $y \rightarrow +\infty$ , and

$$d = \begin{cases} d_k \cos(\frac{k-2}{2} \frac{\pi}{k}) & \text{for even } k, \\ d_k \cos(\lfloor \frac{k-2}{2} \rfloor \frac{\pi}{k} + \frac{\pi}{2k}) & \text{for odd } k, \end{cases}$$

as  $y \rightarrow -\infty$ . Here the weight  $\rho^*(y)$ , is defined as

$$\rho^*(y) = e^{-a|y|^{\frac{2k+1}{2k}}} \quad \text{for all } |y| \geq 1,$$

for  $a \in (0, 2d)$ .

## 2.6 Estimates on the Fundamental Kernel, Majorizing Operator, Spectral Properties, and Comparison

### 2.6.1 Estimates

Recalling our estimate given by (2.9), we now look to estimate our rescaled fundamental kernel  $F(y)$  by

$$|F(y)| \leq \bar{D}\bar{F}(y), \quad \text{where } \bar{F}(y) > 0 \quad \text{and} \quad \int \bar{F}(y) = 1. \quad (2.50)$$

Here  $\bar{D}$  is a normalisation constant, obviously satisfying  $\bar{D} > 1$ . There exists infinitely many functions which satisfy (2.50), but since our kernel  $F$  is changing sign, it is not possible to find an optimal analytic function

$$\bar{F}_{\text{opt}}(y) = \omega_1 |F(y)|,$$

where  $\omega_1 > 0$  is a normalisation constant, such that

$$\int \bar{F}_{\text{opt}} = 1.$$

In this case

$$\omega_1 = \left( \int |F| \right)^{-1} > 1,$$

since  $\int F = 1$ . However, we can find an analytical approximation of  $\bar{F}_{\text{opt}}$  such that (2.50) is satisfied, but is non-optimal. One such function is given by

$$\bar{F}_*(y) = \omega_1 (1 + y^2)^{-\frac{(2k-1)}{8k}} \left( \frac{1}{1+e^{-y}} + \frac{1}{1+e^y} e^{-a(1+y^2)^{\frac{\alpha}{2}}} \right).$$

A sketch of the function  $\bar{F}_*$  is shown by Figure 2.6.1 and comparison with the numerics in Section 2.1.2 shows how the function may be an upper bound for  $F(y)$ .

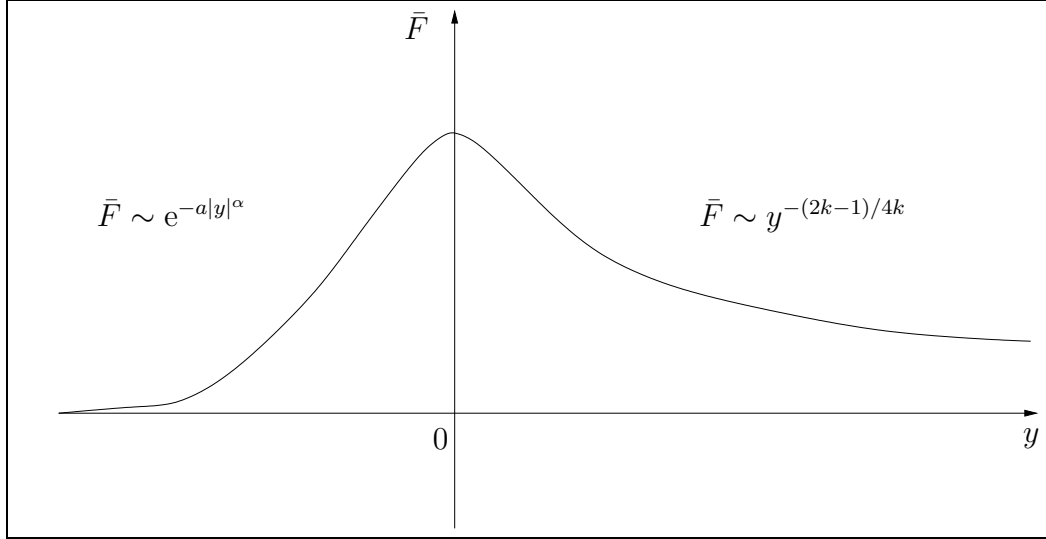


Figure 2-3: Majorizing kernel  $\bar{F}(y)$ .

By using the asymptotic analysis set out in Section 2.1.1, we can also find

that all higher order derivatives are estimated by

$$|D^\beta F(y)| \leq \begin{cases} \bar{c}^\beta \beta^{\frac{(\alpha-1)}{\alpha}} e^{-\bar{a}|y|^\alpha} & \text{for } y \leq -1, \\ \bar{c}^\beta y^{\frac{\beta}{2k}} & \text{for } y \geq 1, \end{cases} \quad (2.51)$$

where  $\bar{c}$  is dependent on  $k$  only, and where  $\bar{a}$  is slightly smaller than  $a$ , but not essential.

## 2.6.2 Majorizing Kernel and Spectral Properties

We now introduce the positive majorizing kernel

$$\bar{b}(x, t) = t^{-\frac{1}{2k+1}} \bar{F}(y), \quad y = xt^{-\frac{1}{2k+1}},$$

which is majorizing relative to the kernel  $b(x, t)$ .

We end up with a formal equation written in the standard form

$$\bar{u}_t = \bar{\mathbf{A}}(t)\bar{u}, \quad (2.52)$$

for some linear operator  $\bar{\mathbf{A}}(t)$ , with the “fundamental solution”  $\bar{b}(x, t)$ . This formal non-autonomous (in time  $t$ ) evolution equation is understood in the sense that the Cauchy problem for (2.52) with initial data

$$\bar{u}(x, 0) = \bar{u}_0(x) \geq 0 \quad \text{in } \mathbb{R}, \quad (2.53)$$

is given by the convolution

$$\bar{u}(x, t) = \bar{\mathbf{M}}(t)\bar{u}_0(x) \equiv \bar{b}(t) * \bar{u}_0 = t^{-\frac{1}{2k+1}} \int \bar{F}((x-z)t^{-\frac{1}{2k+1}}) \bar{u}_0(z) dz. \quad (2.54)$$

It can be seen from this, that for general kernels  $\bar{F}$ , majorizing semigroups do not exist, so equation (2.52) does not admit translation in time. This defines the corresponding majorizing integral equation. For higher-order parabolic (polyharmonic) equation, the idea of majorizing integral operators was introduced and applied in blow-up, studied in [26].

As before, let us now introduce rescaled variables

$$\bar{u}(x, t) = t^{-\frac{1}{2k+1}} \bar{w}(y, \tau), \quad y = xt^{-\frac{1}{2k+1}}, \quad \tau = \ln t.$$

Hence from convolution in (2.54)

$$\bar{w}(y, \tau) \equiv \int \bar{F}(y - ze^{-\frac{\tau}{2k+1}}) \bar{u}_0(z) dz.$$

Using Taylor's Power Series we have that

$$\begin{aligned} \bar{F}\left(y - ze^{-\frac{\tau}{2k+1}}\right) &= \sum_{(\beta)} e^{-\frac{\beta\tau}{2k+1}} \frac{(-1)^\beta}{\beta!} D_y^\beta \bar{F}(y) z^\beta \\ &\equiv \sum_{(\beta)} e^{-\frac{\beta\tau}{2k+1}} \frac{1}{\sqrt{\beta!}} \bar{\psi}_\beta(y) z^\beta, \end{aligned} \tag{2.55}$$

where

$$\bar{\psi}_\beta(y) = \frac{(-1)^\beta}{\sqrt{\beta!}} D_y^\beta \bar{F}(y).$$

The convergence of (2.55) on bounded intervals is guaranteed by the estimates of  $D_y^\beta \bar{F}(y)$  given in (2.51).

The solution can then be represented by

$$\bar{w}(y, \tau) = \sum_{(\beta)} e^{-\frac{\beta\tau}{2k+1}} \bar{M}_\beta(u_0) \bar{\psi}_\beta(y),$$

where we define  $\bar{\lambda}_\beta = -\frac{\beta}{2k+1}$  and

$$\bar{M}_\beta(\bar{u}_0) = \frac{1}{\sqrt{\beta!}} \int_{\mathbb{R}} z^\beta \bar{u}_0(z) dz,$$

are the corresponding momenta of the initial data.

**Proposition 2.6.1** *There exists some formal operator  $\bar{\mathbf{B}}$ , such that*

$$\bar{w}_\tau = \bar{\mathbf{B}} \bar{w}$$

*and this induces the majorizing semigroup  $\{e^{\bar{\mathbf{B}}\tau}\}$ , formulated by the rescaled variables.*

It follows from (2.18) that  $\bar{\mathbf{B}}$  has point spectrum given by

$$\sigma_p(\bar{\mathbf{B}}) = \{\bar{\lambda}_\beta\},$$

and corresponding eigenfunctions are thus given by  $\{\bar{\psi}_\beta\}$ .

In order to find the explicit form of the majorizing semigroup however we once again perform another rescaling given by

$$\bar{u} = (1+t)^{-\frac{1}{2k+1}} \bar{w}, \quad y = x(1+t)^{-\frac{1}{2k+1}}, \quad \tau = \ln(1+t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+.$$

Then rescaling the convolution we obtain

$$\begin{aligned} \bar{w}(y, \tau) &= e^{\bar{\mathbf{B}}\tau} \bar{u}_0 \\ &\equiv (1 - e^{-\tau})^{-\frac{1}{2k+1}} \int_{\mathbb{R}} \bar{F} \left( (y - ze^{-\frac{\tau}{2k+1}})(1 - e^{-\tau})^{-\frac{1}{2k+1}} \right) \bar{w}_0(z) dz. \end{aligned}$$

Using Taylor expansions we find the solution can be given by

$$\begin{aligned} \bar{w}(y, \tau) &= (1 - e^{-\tau})^{-\frac{1}{2k+1}} \sum_{(\mu, \nu)} \frac{(-1)^\mu}{\mu! \nu!} D^\nu \bar{F}(0) \frac{1}{(\nu - \mu)!} y^{\nu - \mu} (e^\tau - 1)^{-\frac{\mu}{2k+1}} \\ &\quad \times (1 - e^{-\tau})^{-\frac{\nu}{2k+1}} \int_{\mathbb{R}} z^\mu \bar{w}_0(z) dz. \end{aligned}$$

The adjoint operator  $\bar{\mathbf{B}}^*$  with polynomial eigenfunctions  $\{\bar{\psi}_\beta^*\}$  occurs if we use the blow-up scaling

$$\bar{u}(x, t) = \bar{w}(y, \tau), \quad y = x(1-t)^{-\frac{1}{2k+1}}, \quad \tau = -\ln(1-t).$$

We thus obtain

$$\bar{w}(y, \tau) = (1 - e^{-\tau}) \int_{\mathbb{R}} \bar{F} \left( (ye^{-\frac{\tau}{2k+1}} - z)(1 - e^{-\tau})^{-\frac{1}{2k+1}} \right) \bar{w}_0(z) dz.$$

Using Taylor expansion yields

$$\begin{aligned} \bar{w}(y, \tau) &= (1 - e^{-\tau})^{-\frac{1}{2k+1}} \sum_{(\beta, \nu)} \frac{(-1)^\beta}{\beta! \nu!} (1 - e^{-\tau})^{-\frac{\beta}{2k+1}} (e^\tau - 1)^{-\frac{\nu}{2k+1}} \\ &\quad \times D^\nu F(0) \frac{1}{(\nu - \beta)!} y^{\nu - \beta} \int_{\mathbb{R}} z^\beta \bar{w}_0(z) dz. \end{aligned}$$

As above, further expanding of exponential terms here, leads to the eigenfunction expansion, which determines finite generalised Hermite polynomials. This representation is rather technical and we do not present and analyze it, since we do not aim to use those polynomials in what follows. Our main application of



the majorizing operators is as follows:

### 2.6.3 Comparison with Majorizing Problem

We look to see how the majorizing kernel relates to our real solution  $u(x, t)$ . By looking at the convolution (2.12) for the linear PDE, we can easily see that

$$|u(x, t)| \leq |b(t)| * |u_0(x)|.$$

Now looking at our estimate of the majorizing kernel,  $\bar{b}(x, t)$ , we have that

$$|u(x, t)| \leq \bar{D}\bar{b}(x, t) * |u_0(x)| \leq \bar{b}(t) * \bar{u}_0(x),$$

where we have to assume the following inequality for initial data:

$$\bar{D}|u_0(x)| \leq \bar{u}_0(x) \quad \text{in } \mathbb{R}. \quad (2.56)$$

**Proposition 2.6.2** *If (2.53) and (2.56) hold, then*

$$|u(x, t)| \leq \bar{u}(x, t) \quad \text{in } \mathbb{R} \times \mathbb{R}_+.$$

We introduce the majorizing kernel, since the structural behaviour of self-similar solutions of the majorizing evolution equation describes essential features of the solutions to the original PDE. Since we have estimated our solution, it can be possible to find properties of the real solution, which otherwise would be more difficult to achieve.

The linear semigroup for  $\mathbf{B}$  is not order-preserving, since the kernel  $F$  is oscillatory. Therefore we must have  $\bar{D} > 1$ , which gives the order deficiency of the linear operator  $\mathbf{B}$  and of the linear convolution operator  $\bar{M}(t)$ , given in (2.54). The actual defect is actually characterised by  $\bar{D} - 1 > 0$ . The linear semigroup for  $\mathbf{B}$  would only be order-preserving if  $F$  did not change sign, since then we would have that  $F = \bar{F}$  and so we would have  $\bar{D} = 1$ , hence the defect would be zero. As we have shown, this is not possible for any  $k \geq 1$ .

# Chapter 3

## Semilinear Dispersion PDEs

We now consider the odd-order problem, but now with a nonlinear absorption. We look at the Cauchy problem for the semilinear odd-order equation

$$u_t = (-1)^{k+1} D_x^{2k+1} u - u^p \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad (3.1)$$

with initial data  $u(x, 0) = u_0(x)$ , for  $k = 1, 2, 3, \dots$ . Here  $p > 1$  is a fixed absorption exponent. For convenience, we write

$$u^p := |u|^{p-1}u,$$

in order to avoid any singularities due to the power  $p$ , which may be caused by the changing sign of the solution  $u(x, t)$ . This model is connected to the KdV equation (1.2), with the difference being that the extra operator  $-u^p$ , which corresponds to absorption, is simpler and of zero differential order. This kind of nonlinearity also allows to avoid to enter these classes of integrable PDEs, which obeys various specific features that are illusive for more general equations.

As with the linear case, even-order semilinear problems have been studied over recent years and are fairly well understood, unlike similar odd-order ones. The related generalised semilinear even-order model, is given by

$$u_t = -(-\Delta)^m u \pm |u|^{p-1}u, \quad (3.2)$$

which includes both the cases of absorption and reaction. Obviously the lower order absorption case, with  $m = 2$ , corresponds to the heat equation with absorption, given in (1.9).

Various papers have been written on the parabolic semilinear model (3.2), which include [12, 17, 20, 22, 23, 30], amongst others. We follow some of these methods and apply them to the odd-order case.

### 3.1 Similarity Solutions of Semilinear Equations

As is customary in PDE theory, there exists a critical exponent for (3.1) given by

$$p = p_0 = 1 + \frac{2k+1}{N} = 2k + 2, \quad (3.3)$$

for  $N = 1$ . It can be called the critical *Fujita* exponent; see further comments below. As for the parabolic equation (3.2), where

$$p_0 = 1 + \frac{2m}{N}$$

(as in (3.3),  $2m$  stands for the order of the differential operator involved), the critical Fujita exponent characterises parameter ranges of blow-up and non-blow-up solutions and changing of the stability of the trivial zero solutions, for the PDEs under consideration.

As usual we consider self-similar solutions of the very singular type of (3.1)

$$u_*(x, t) = t^{-\frac{1}{p-1}} f(y), \quad y = xt^{-\frac{1}{2k+1}},$$

where  $f$  solves the ODE

$$(-1)^{k+1} f^{(2k+1)} + \frac{1}{2k+1} f' y + \frac{1}{p-1} f - |f|^{p-1} f = 0 \quad \text{in } \mathbb{R}. \quad (3.4)$$

Unlike the linear case, we cannot integrate in order to reduce the order of the ODE and hence we remain with an equation of order  $2k + 1$ . We see that (3.4) is a difficult higher-order equation and so begin with numerical results.

## 3.2 Numerical Results for the Semilinear Equation

We look at similarity profiles of the semilinear problem (3.4), for  $k = 1$ ,

$$f''' + \frac{1}{3}f'y + \frac{1}{p-1}f - f^p = 0 \quad \text{in } \mathbb{R},$$

where, as we have said,  $f^p = |f|^{p-1}f$ .

For computations where there is a nonlinearity in the rescaled solution, it is important to look at a *regularised* version of the ODE. Here the regularised equation is given by

$$f''' + \frac{1}{3}f'y + \frac{1}{p-1}f - (f^2 + \epsilon^2)^{\frac{p-1}{2}}f = 0,$$

for some small  $\epsilon > 0$ . The value of  $\epsilon$  does not matter too much, as long as it is small enough not to affect the solutions, but large enough not to be completely negligible. Typically we take  $\epsilon = 10^{-4}$  or smaller.

Once again the Matlab `bvp4` solver is used, to plot the profiles for the rescaled solutions. This however gives us the same problem, as in Section 2.1.2, of solving an initial value problem using the Boundary Value Problem solver. We take  $f$  and  $f'$  to be zero, as boundary conditions on the left side and  $f$  to be zero, as the condition on the right-hand side. We then try to shift the point at the right boundary, in order to coincide with a point where the oscillations go through zero and hence find the best profile. We look in particular for convergence and reflectional symmetry of the tail.

Figures 3-3 to 3-6 show four sets of profiles in the range of

$$1.9 \leq p \leq 3.3.$$

We also refer to Appendix A, for more profiles with values in this range. Whilst we cannot guarantee the accuracy of the size of the tail, the profile close to the origin, where  $\max |f|$  is obtained, is very stable. We see that as  $p$  decreases,  $\max |f|$  increases and this seems to justify the term “Very Singular”, with the mass concentrated close to  $y = 0$ .

In the linear case, set out in Section 2.1.2, due to the instability of the solutions, we had to use a “matching technique”, in order to find reliable profiles.

However in the semilinear case this is not necessary, as the profiles are stable, since we do not have a scaling group of solutions  $\{cf\}$ , due to the nonlinear term  $|f|^{p-1}f$ . In fact we find that whilst the tail of the profile may differ, given different boundary points, the rest of the structure is extremely stable and rarely changes. For

$$2 \lesssim p \lesssim 3,$$

we can find profiles easily, for almost every boundary point value we use and they only really differ from the tail. But for other values it is more difficult to find profiles, especially reliable ones.

In particular, we look to see the behaviour of the solutions as  $p \rightarrow 1^+$  and as  $p \rightarrow 4^-$ , which is the critical exponent for  $k = 1$ .

### 3.2.1 Finding Reliable Profiles

Since we are using the boundary value solver on Matlab, we face numerous difficulties in obtaining the correct numerical results. As we are solving the initial value problem of the semilinear equation as a boundary value problem, difficulties arise in finding the correct end-point.

We do not know exactly at which points  $f = 0$ , so we attempt to approximate the point by looking at various profiles and finding the best. If the far right boundary point is incorrect, then we produce artificial oscillations. We know which profiles are most likely to be false, from the analysis we have done on odd-order linear PDEs. Since the semilinear equation is the same as the linear equation with a perturbation of  $|f|^{p-1}f$ , we expect similar behaviour. With a wrong boundary point, we can end up with non-symmetry, since we are forcing the oscillations through a specific point. Hence, the most important condition we look for in reliable profiles is that the oscillations are symmetric.

We also look to ensure that the oscillations become symmetric as quickly as possible. This also follows from having forced oscillations, due to wrong boundary points. Finally, we look for oscillations that decay as  $y$  increases, since this behaviour is known for the LDEs. Certainly we would not expect to have divergent behaviour, as  $y \rightarrow +\infty$ .

The solver that is used requires an initial guess for the function. Whilst this guess does not necessarily have to be very accurate, occasionally a wrong guess can lead to wrong profiles being found. We have yet to find any evidence so far,

to suggest that the initial guess is bad enough to affect the profiles. However for less stable profiles where  $p \lesssim 2$ , the solutions are large and the initial guess for  $\max |f|$  may not be accurate. The values needed for the initial guess for the maximum value are much larger than the maximum value found in the profile. Whilst the profiles look reliable, it is unclear as to whether the inaccurate initial guess affects the output.

We also refer to Figures A-1 to A-4, in Appendix A, which show just a few examples of unreliable profiles when a wrong boundary point has been placed.

Thus the conditions we look at, to ensure the most reliable profiles, can be briefly summarised as follows:

- Symmetry (reflectional,  $f \mapsto -f$ ) of tail for  $y \gg 1$ .
- Symmetry of tail occurs as close to 0 as possible.
- Minimisation of symmetric tail.

In some cases, there may be slight violation of symmetry and also a slight divergence of the tail, but these profiles are the most reliable found, under the conditions set. In particular, this occurs towards the limits  $p \rightarrow 1^+$  and  $p \rightarrow 4^-$ , where numerics are more difficult. It may be possible to improve the reliability of these profiles, but we would need to look at moving the end boundary point at even smaller increments, without guarantee of improvement.

Figures 3-1 and 3-2 show how changing the end boundary point by a small increment, can change the profile. In this case we have taken  $p = 2.9$  and look at end boundary points  $y = 298.8$  and  $y = 298.9$ . It is seen that by moving the boundary point by just 0.1 increases the tail by a large amount.

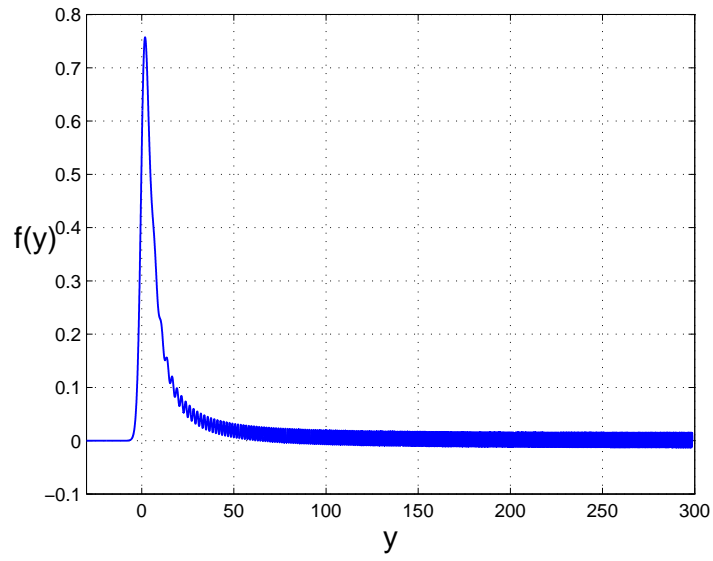


Figure 3-1: VSS profile to  $u_t = u_{xxx} - u^p$ , with  $p = 2.9$ , calculated in the range  $y = [-30, 298.8]$ .

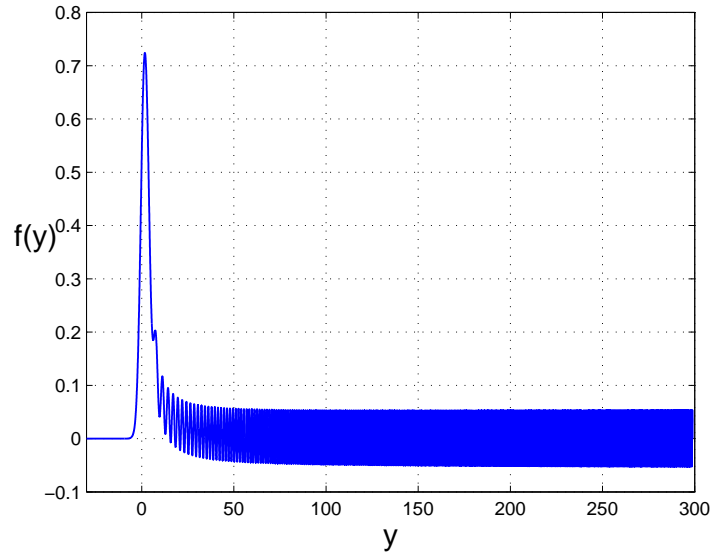
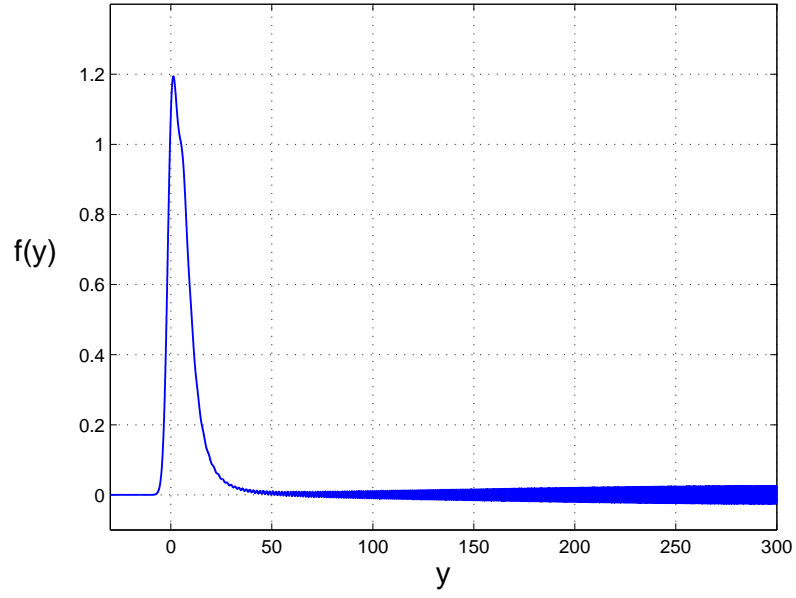
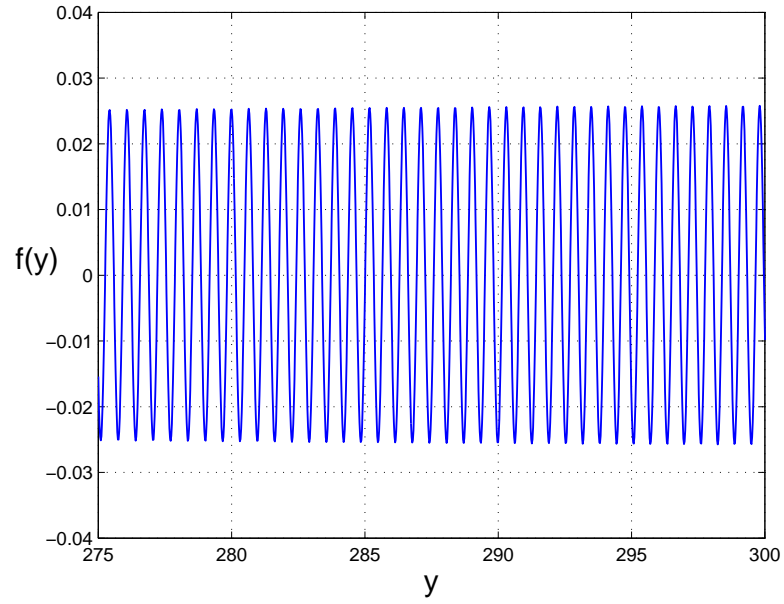


Figure 3-2: VSS profile to  $u_t = u_{xxx} - u^p$ , with  $p = 2.9$ , calculated in the range  $y = [-30, 298.9]$ .



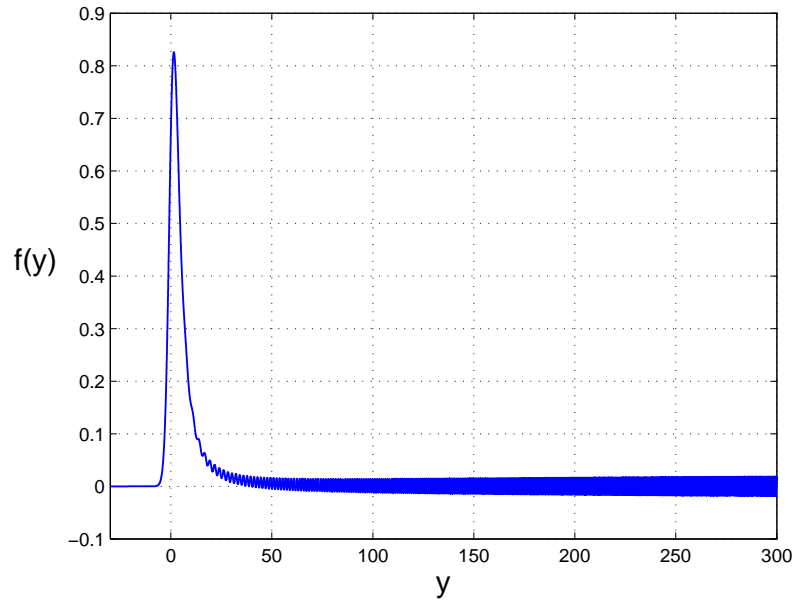
(a) Global solution structure.



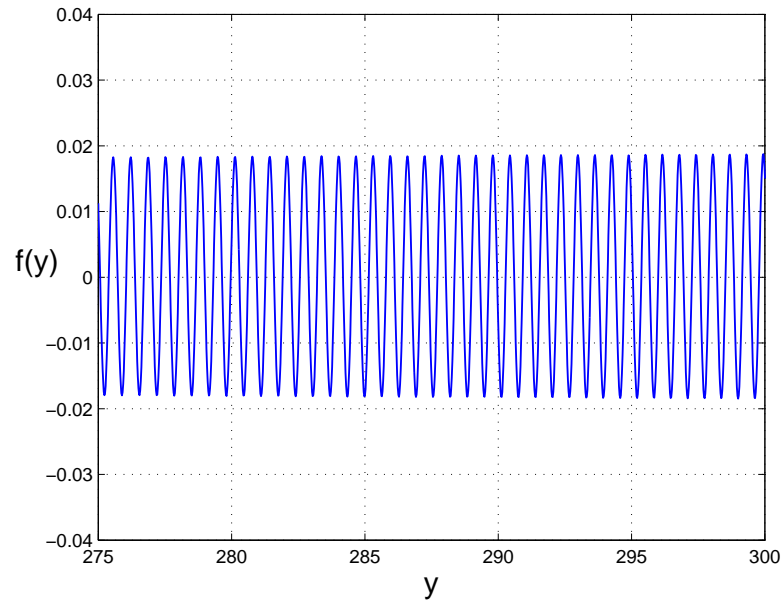
(b) "Tail" of the solution, for  $y \gg 1$ .

Figure 3-3: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 1.9$ .



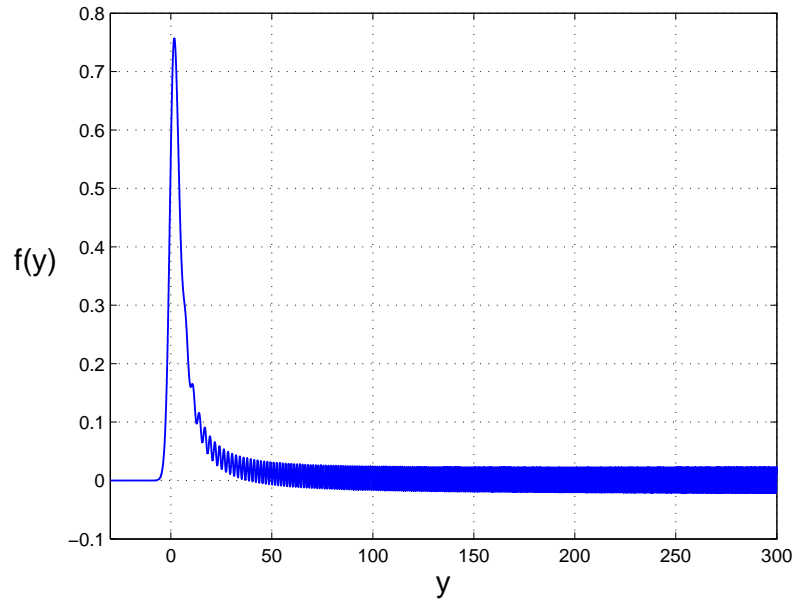


(a) Global solution structure.

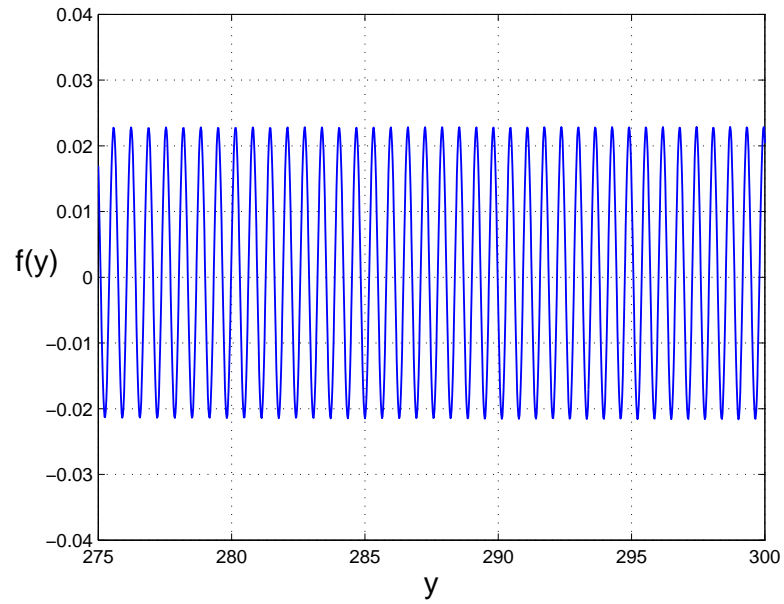


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure 3-4: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.4$ .

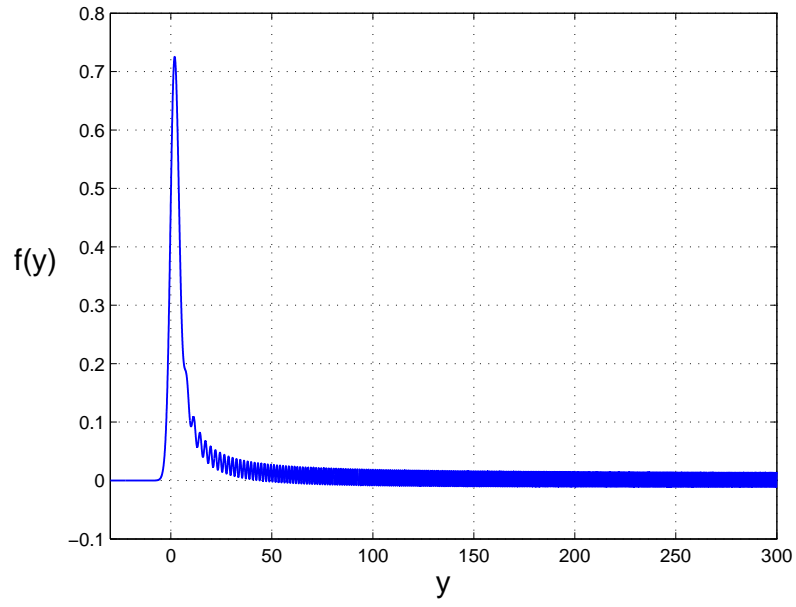


(a) Global solution structure.

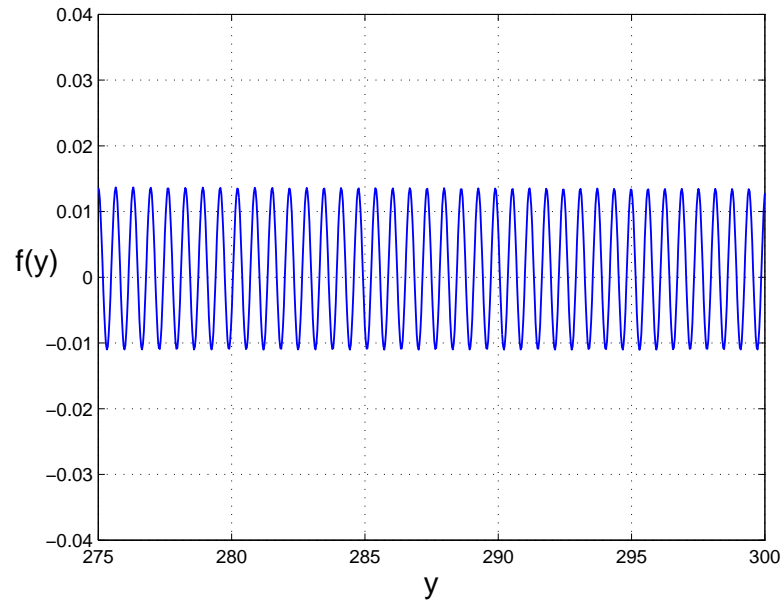


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure 3-5: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.8$ .



(a) Global solution structure.



(b) "Tail" of the solution, for  $y \gg 1$ .

Figure 3-6: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 3.3$ .

### 3.3 Linearised Stability Analysis

We begin with some applications of the spectral analysis obtained before, in the theory for LDEs. In view of the highly oscillatory character of the eigenfunctions  $\{\psi_\beta\}$  of  $\mathbf{B}$ , with no exponential decay, some functional formalities of “dual” space  $\bar{L}^2(\mathbb{R})$ , with the indefinite metric (2.29), are not easy at all and still remain obscure. So some of our future conclusions will be formal and we will need to clearly indicate which ones are.

As in the linear case, let us now introduce the following similarity scaling in the semilinear equation (3.1), with

$$u(x, t) = (1 + t)^{-\frac{1}{2k+1}} v(y, \tau), \quad y = x(1 + t)^{-\frac{1}{2k+1}}, \quad \tau = \ln(1 + t).$$

Our rescaled equation is then given by equation

$$\begin{aligned} v_\tau &= (-1)^{k+1} D_y^{2k+1} v + \frac{1}{2k+1} y D_y v + \frac{1}{p-1} v - |v|^{p-1} v \\ &\equiv \mathbf{B}_1 v - |v|^{p-1} v. \end{aligned} \tag{3.5}$$

Here the linear operator  $\mathbf{B}_1$  is defined by

$$\mathbf{B}_1 = \mathbf{B} + d_1 I, \quad \text{where} \quad d_1 = \frac{1}{p-1} - \frac{1}{2k+1} = \frac{p_0 - p}{(2k+1)(p-1)},$$

and where the operator  $\mathbf{B}$  is from the linear theory in Section 2.3 and  $p_0$  is our critical exponent, (3.3).

**Lemma 3.3.1** *For  $p > p_0 = 2k + 2$ , zero is exponentially linearly stable for (3.5) in  $H_\rho^{2k+1}(\mathbb{R})$ .*

**Proof.** We look at the linearised problem of (3.5) about zero,

$$v_\tau = \mathbf{B}_1 v.$$

Then the spectrum is given by

$$\sigma_p(\mathbf{B}_1) = \left\{ d_1 - \frac{l}{2k+1} \quad \text{for} \quad l \geq 0 \right\}.$$

Therefore for  $l = 0$ ,

$$\lambda_0 = d_1 \equiv \frac{1}{p-1} - \frac{1}{2k+1} < 0, \quad \text{for} \quad p > 2k + 2.$$

So in our space

$$\|v(\tau)\|_{2k+1,\rho} \sim e^{\lambda_0\tau} \rightarrow 0 \quad \text{as } \tau \rightarrow \infty.$$

Hence zero is linearly stable for  $p > p_0$ .  $\square$

**Lemma 3.3.2** *For  $1 < p < p_0$ , zero is exponentially linearly unstable for (3.5), for small data  $v_0 \in H_\rho^{2k+1}$ .*

**Proof.** For  $l = 0$ ,

$$\lambda_0 = d_1 \equiv \frac{1}{p-1} - \frac{1}{2k+1} > 0,$$

so that

$$\|v(\tau)\|_{2k+1,\rho} \sim e^{\lambda_0\tau} \rightarrow \infty.$$

Hence zero is linearly unstable for  $p < p_0$ .  $\square$

## 3.4 Centre Subspace Behaviour

### 3.4.1 First Critical Exponent $p_0$

We look again at the rescaled equation given by (3.5). Let us look at the case where  $l = 0$ , with critical exponent  $p = p_0 = 2k + 2$ . We check the behaviour close to the centre subspace of  $\mathbf{B}$ , i.e., we set

$$v(\tau) = c_0(\tau)\psi_0 + v_0^\perp(\tau),$$

where  $v_0^\perp$  is asymptotically small in comparison with the first term and orthogonal to  $\psi_0$ , i.e.,  $\langle v_0^\perp, \psi_0^* \rangle_* = 0$ . Then since  $\mathbf{B}\psi_0 = 0$ , we multiply by  $\psi_0^* \equiv 1$  to get the following leading term:

$$c_0' = -c_0^{2k+2} \langle |\psi_0|^{2k+1} \psi_0 + \dots, \psi_0^* \rangle_*.$$

We now let

$$\gamma_0 = \langle |\psi_0|^{2k+1} \psi_0, \psi_0^* \rangle_* \neq 0,$$

for convenience. Note that analytically, proving that  $\gamma_0 \neq 0$  is very difficult and even checking this numerically is also questionable. So

$$c_0^{-(2k+2)} c_0' = -\gamma_0 + \dots \quad \text{as } \tau \rightarrow \infty,$$

and therefore

$$-\frac{1}{2k+1}c_0^{-(2k+1)} = -\gamma_0\tau + \dots$$

Finally, this yields the following rate of decay

$$c_0(\tau) \approx [(2k+1)\gamma_0\tau]^{-\frac{1}{2k+1}}, \quad \text{for } \tau \gg 1.$$

So the centre subspace behaviour of  $u(x, t)$  is given, as  $t \rightarrow \infty$ , by

$$u(x, t) \approx (1+t)^{-\frac{1}{2k+1}} [(2k+1)\gamma_0 \ln(1+t)]^{-\frac{1}{2k+1}} \psi_0(x(1+t)^{-\frac{1}{2k+1}}),$$

i.e., contains a typical extra logarithmic factor. A full justification of such a behaviour remains open.

### 3.4.2 Other Critical Exponents $p_l$ : Stable Subspace Behaviour

Let us now look at the general case for  $l$ , with critical point  $p = p_l = 1 + \frac{2k+1}{l+1}$ , where we check the behaviour close to the one-dimensional kernel of  $\mathbf{B} - \lambda_l I$ , by setting

$$v = c_l(\tau)\psi_l + v_l^\perp.$$

Here,  $v_l^\perp$  is small and orthogonal to  $\psi_l$ , as before. Then for  $\mathbf{B}\psi_l = \lambda_l\psi_l$ , we take the scalar product with  $\psi_l^*$  to get

$$c'_l \approx -c_l^{\frac{2k+2+l}{l+1}} \langle |\psi_l|^{\frac{2k+1}{l+1}} \psi_l + \dots, \psi_l^* \rangle_*.$$

For convenience, let as usual

$$\gamma_l = \langle |\psi_l|^{\frac{2k+l}{l+1}} \psi_l, \psi_l^* \rangle_* \quad (\neq 0).$$

Hence assuming  $\gamma_l > 0$

$$c_l^{-\frac{2k+2+l}{l+1}} c'_l = -\gamma_l + \dots \quad \text{as } \tau \rightarrow \infty,$$

therefore

$$-\frac{l+1}{2k+1} c_l^{-\frac{2k+1}{l+1}} = -\gamma_l \tau + \dots,$$

so that

$$c_l(\tau) \approx \left(\frac{2k+1}{l+1}\gamma_l\tau\right)^{-\frac{l+1}{2k+1}} \quad \text{for } \tau \gg 1.$$

Hence the stable subspace behaviour of  $u(x, t)$  as  $t \rightarrow \infty$ , for all critical points, can be given by

$$u(x, t) \approx (1+t)^{-\frac{1}{2k+1}} \left[\frac{2k+1}{l+1}\gamma_l \ln(1+t)\right]^{-\frac{l+1}{2k+1}} \psi_l(x(1+t)^{-\frac{1}{2k+1}}).$$

So there exists a countable set of stable subspace behaviours governed by eigenfunctions, corresponding to the point spectrum  $\sigma(\mathbf{B}) = \{-\frac{l}{2k+1}, \quad l \geq 0\}$ .

**Remark: Why  $\gamma_l > 0$  :**

For the above analysis to hold, we need to show that

$$\gamma_l = \langle |\psi_l|^{\frac{2k+1+l}{l+1}} \psi_l, \psi_l^* \rangle_* > 0,$$

which is not an easy inequality to prove.

Let us first look at the case  $l = 0$ . We need to show that

$$\gamma_0 = \langle |\psi_0|^{2k+1} \psi_0, \psi_0^* \rangle_* > 0.$$

However in this case we have that  $\psi_0^* \equiv 1$ , so in essence it is enough to prove that

$$\gamma_0 = \int |\psi_0|^{2k+1} \psi_0 > 0.$$

Attempting to prove this rigorously, is very difficult as well. However, numerically it can be shown, for  $k = 1$  at least, that this inequality is true.

Certainly we know that the solution  $F$  can be found, for the lower order case  $k = 1$ . Hence using the Matlab function *trapz*, which uses a trapezoidal method of integration, we can solve. Using this method, the integral can be approximated to

$$\int |F|^3 F = 0.0300 \dots > 0. \tag{3.6}$$

It is noted that since this numerical solution  $F$ , is not the fundamental kernel satisfying  $\int F = 1$ , we scale the calculations such that this is true. So (3.6) holds for the true rescaled fundamental kernel.

Solving for  $k > 1$  is much more difficult as the shooting problem to solve  $F$  is not easy, but not impossible.

Similarly, it is possible to construct solutions for all  $\psi_l$ , where  $l \geq 0$ , using conservation laws. Again the problem of shooting is difficult, but not impossible to prove the inequality.

### 3.5 Bifurcation Points

We follow classic ideas of bifurcation theory to formally analyse critical points  $p_l$ , and in particular the *Lyapunov-Schmidt* reduction is used; see [45, 41].

Once again, we look at our semilinear equation given in (3.4), which we can now write as

$$\mathbf{B}f + \left(\frac{1}{p-1} - \frac{1}{2k+1}\right)f - |f|^{p-1}f = 0 \iff \mathbf{B}_1f - |f|^{p-1}f = 0,$$

where  $\mathbf{B}_1 = \mathbf{B} + d_1I$  is defined as before.

Let us look at our critical points, which have been used in previous analysis. Critical values occur when

$$d_1 \equiv \frac{1}{p_l-1} - \frac{1}{2k+1} = -\lambda_l,$$

i.e., when

$$\frac{1}{p_l-1} - \frac{1}{2k+1} = \frac{l}{2k+1}.$$

Therefore our critical exponents,  $p_l$ , are given by

$$p_l = 1 + \frac{2k+1}{l+1}, \quad l = 0, 1, 2, \dots$$

We can see from this that  $p_l \rightarrow 1^+$  as  $l \rightarrow +\infty$ .

We look at values of  $p$  near these critical values, so that  $p \approx p_l$ . We set  $\epsilon = p_l - p$ , then

$$(\mathbf{B} - \lambda_l I)f + \epsilon a_0 f = |f|^{p-1}f + O(\epsilon^2), \quad (3.7)$$

where  $a_0$  is some constant. We can find  $a_0$ , by substituting in  $\epsilon$  and so we have

$$a_0 = \frac{1}{(p_l-1)^2} = \left(\frac{l+1}{2k+1}\right)^2.$$



Our solution  $f$  can be given by

$$f = C\psi_l + w^\perp,$$

where  $w^\perp$  is orthogonal to  $\psi_l$ ,

$$\langle w^\perp, \psi_l^* \rangle_* = 0.$$

Thus, taking the scalar product of (3.7), in the indefinite metric, with  $\psi_l^*$ , we have that

$$a_0 \epsilon C \langle \psi_l, \psi_l^* \rangle_* = |C|^{p-1} C \langle |\psi_l|^{p-1} \psi_l, \psi_l^* \rangle_*.$$

Since  $\langle \psi_l, \psi_l^* \rangle_* = 1$ , we find

$$|C|^{p-1} = \frac{a_0 \epsilon}{\langle |\psi_l|^{p-1} \psi_l, \psi_l^* \rangle_*} = \frac{1}{\kappa_l} \left( \frac{l+1}{2k+1} \right)^2 \epsilon,$$

where

$$\kappa_l = \langle |\psi_l|^{p-1} \psi_l, \psi_l^* \rangle_*.$$

For  $p = 1$ , we have that

$$\kappa_l = \langle \psi_l, \psi_l^* \rangle_* = 1$$

and so by continuity with respect to  $p$ , we must have that

$$\kappa_l > 0 \quad \text{for all } p \approx 1^+.$$

We therefore expect a *subcritical pitchfork* bifurcation, though proving this is extremely difficult.

Figure 3.5 shows a numerical calculation for the first branch of the bifurcation diagram, where we take  $l = 0$  and  $k = 1$ . Hence, in this case, the critical point is  $p = 4$ . During each iteration of the numerical program, the calculation uses the previous results to calculate the next step, thus improving the accuracy. The step size used here is 0.001, in the range of  $p = 1.7$  to  $p = 3.3$ . Extending the range of values of  $p$  proves to be difficult. Indeed, as shown in Section 3.2, finding numerics outside the range of  $2 \lesssim p \lesssim 3$ , has proven to be difficult.

Figure 3.5 shows how the bifurcation diagram for the semilinear equation is expected to look like, if the branch is extended, given the numerics in Figure 3.5.

However the numerical analysis done in Section 3.2 has failed to provide firm

evidence that Figure 3.5 shows how the  $p$ -branches behave, since we do not have reliable numerics close to  $p = 4$ . As before, this is related to the extremal oscillatory behaviour of similarity profiles at the right-hand side, which does not allow us using standard numerical codes of continuation with respect to the parameter  $p$ , i.e., numerically construct the so-called  $p$ -branches of solutions. In Figure 3.5, we use the analytical evidence of such bifurcations from zero at  $p = p_l$ , which also requires extra very difficult mathematical justification.

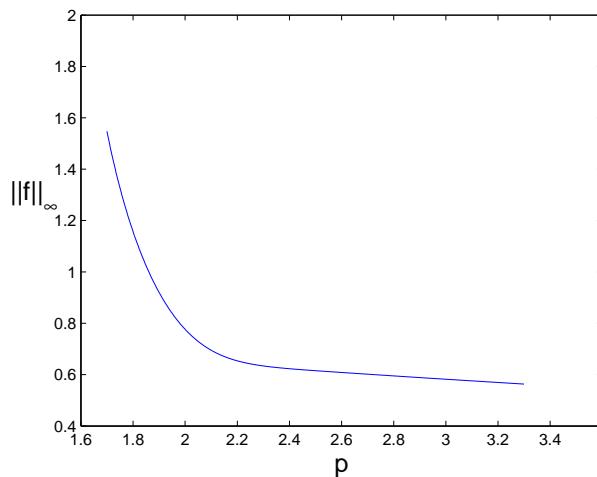


Figure 3-7: Bifurcation branch for  $l = 0$  and  $k = 1$ , for the semilinear ODE (3.4).

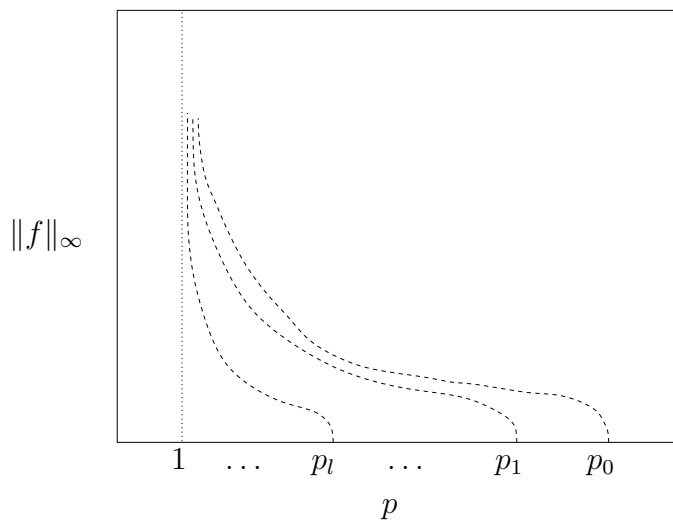


Figure 3-8: Expected bifurcation in  $p$  for the semilinear ODE (3.4).

# Chapter 4

## Nonlinear Dispersion PDEs

We now look at *nonlinear dispersion equations* (NDEs), which have the general form

$$u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) + \tilde{g}(u), \quad n > 0, \quad (4.1)$$

where  $\tilde{g}(u)$  is some function of  $u$  and may include some differential term with respect to  $x$ .

### 4.1 Nonlinear Models: Quasilinear KdV-type Equations and Parabolic PDEs

First, consider the higher odd-order nonlinear dispersion equation (NDE), given by

$$u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) + (|u|^n u)_x. \quad (4.2)$$

Surprisingly, it can be shown that (4.2) is somehow related to the parabolic even-order equation, which is given by

$$u_t = (-1)^{k+1} D_x^{2k}(|u|^n u) + |u|^n u. \quad (4.3)$$

Equation (4.3) admits blow-up self-similar solutions of the separate form

$$u(x, t) = (T - t)^{-\frac{1}{n}} f(x), \quad (4.4)$$

where  $T$  is the finite blow-up time. This self-similarity reduces the PDE to the

ODE for the similarity profile  $f$ :

$$(-1)^{k+1} D_x^{2k}(|f|^n f) + |f|^n f = \frac{1}{n} f. \quad (4.5)$$

It can be seen that equation (4.5), for  $k = 1$  only, is known to possess the explicitly compactly supported solution

$$f(x) = \left[ \frac{2(n+1)}{n(n+2)} \cos^2 \left( \frac{nx}{2(n+1)} \right) \right]^{\frac{1}{n}}.$$

Therefore, setting  $f(x) \equiv 0$  for all

$$|x| \geq \frac{\pi(n+1)}{n},$$

we obtain from (4.4) the so-called *standing wave blow-up solution* (S-regime of blow-up), which always have compact support; see details in [62, Ch. 4]. For any  $k > 1$ , the ODE (4.5) cannot be solved explicitly.

The lower order case for equation (4.3) (with  $k = 1$ ), which is just a reaction-diffusion PDE, is fairly well understood. However the third order nonlinear dispersion equation in (4.2), for  $k = 1$ , has not been studied extensively and some basic principles are still relatively unknown.

#### 4.1.1 Compactons in NDEs: Compactly Supported Travelling Waves

Equation (4.2) is a generalisation of the third order *Rosenau-Hyman* (RH) *equation*

$$u_t = (u^2)_{xxx} + (u^2)_x,$$

which models the effect of nonlinear dispersion in the pattern formation of liquid drops (see [59]). It can easily be seen that for  $n = 1$  in (4.2), we reduce to the RH equation.

It is known that the RH equation possesses explicit moving compactly supported, soliton-type solutions, known as *compactons*. Compactons have the same structure as travelling wave solutions, given by

$$u_c(x, t) = f(z), \quad z = x - \lambda t.$$

So looking at compacton solutions for (4.2), on substitution we have that

$$-\lambda f' = (-1)^{k+1} D_z^{2k+1}(|f|^n f) + (|f|^n f)'.$$

After integrating once, we find  $f(z)$  satisfies

$$-\lambda f = (-1)^{k+1} D_z^{2k}(|f|^n f) + (|f|^n f). \quad (4.6)$$

For  $k = 1$ , this possesses the exact same travelling wave solution as the nonlinear parabolic equation (4.3), with

$$\lambda = -\frac{1}{n},$$

where  $f$  satisfies (4.5).

Whilst we note that compacton solutions may be found for nonlinear dispersion equations, we do not ourselves apply this method and instead look to find similarity solutions as before.

## 4.2 Similarity Solutions of the NDE

Let us now consider the pure NDE

$$u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \quad (4.7)$$

which is connected to (4.2), but now we do not have the convection-like term  $(|u|^n u)_x$ . Our nonlinear equation may be compared with the even-order model, which represents the general higher-order porous medium equation (PME)

$$u_t = (-1)^{m+1} \Delta^m(|u|^{n-1} u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad m > 1.$$

The PME appears in a number of physical applications, such as fluid flow, heat transfer or diffusion. Other applications have been proposed in mathematical biology, lubrication, boundary layer theory and other fields. For papers exploring the PME, see [24], where further references can also be found.

Our NDE (4.7) has standard similarity solutions given by

$$u_s(x, t) = t^{-\alpha} f(y), \quad y = xt^{-\beta}, \quad (4.8)$$

for some unknown  $\alpha, \beta$ . After substitution into the NDE, we obtain the ODE

$$-\alpha t^{-\alpha-1} f - \beta t^{-\alpha-1} f' y = (-1)^{k+1} t^{-\alpha(n+1)-\beta(2k+1)} D_y^{2k+1}(|f|^n f). \quad (4.9)$$

By equating powers of  $t$ , the parameter  $\beta$  can be found in terms of  $\alpha$  and is given by

$$\beta = \frac{1-\alpha n}{2k+1} > 0,$$

where we see that

$$\alpha < \frac{1}{n} \quad \text{for} \quad \beta > 0.$$

Our ODE (4.9) can then be reduced to

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \alpha f + \frac{1-\alpha n}{2k+1} f' y = 0. \quad (4.10)$$

Unlike previous examples,  $\alpha$  and hence  $\beta$ , cannot be found explicitly from this. Here  $\alpha > 0$  at this stage is still unknown, but will play a role of the “nonlinear eigenvalue”. It can be seen that for  $n = 0$ ,  $\alpha$  corresponds to the eigenvalue  $\lambda_l$  in our linear operator theory.

### 4.2.1 Conservation Laws

It turns out that some fundamental eigenvalues can be calculated explicitly by conservation laws.

Assuming that the solution  $u(x, t)$  is integrable, we have that (4.7) is conservative in mass and so

$$\frac{d}{dt} \int_{\mathbb{R}} u(x, t) dx = 0. \quad (4.11)$$

For similarity solutions (4.8) we have that

$$\int_{\mathbb{R}} u(x, t) dx = t^{\beta-\alpha} \int_{\mathbb{R}} f(y) dy.$$

This satisfies (4.11) if we have that

$$-\alpha + \beta = 0 \implies \alpha = \frac{1}{(2k+1)+n},$$

for non-zero rescaled mass  $\int f \neq 0$ . So on substitution into (4.10),

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \frac{1}{(2k+1)+n} f + \frac{1}{(2k+1)+n} f' y = 0,$$

and integrating once, we end up with the ODE

$$(-1)^{k+1} D_y^{2k}(|f|^n f) + \frac{1}{(2k+1)+n} f y = 0.$$

Note that for  $n = 0$  we have exactly the linear ODE (2.4).

For convenience, we use the natural substitution

$$Y = |f|^n f \implies f = |Y|^{-\frac{n}{n+1}} Y, \quad (4.12)$$

in order to remove any nonlinearities in the highest differential. Substitution yields

$$(-1)^{k+1} D_y^{2k} Y(y) + \frac{1}{(2k+1)+n} y |Y(y)|^{-\frac{n}{n+1}} Y(y) = 0. \quad (4.13)$$

Similarly, we have conservation of the first moment, with

$$\int_{\mathbb{R}} x u(x, t) dx = t^{2\beta-\alpha} \int_{\mathbb{R}} y f(y) dy.$$

Hence we have that

$$\alpha = \frac{2}{(2k+1)+2n}.$$

This then gives the ODE

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \frac{2}{(2k+1)+2n} f + \frac{1}{(2k+1)+2n} f' y = 0. \quad (4.14)$$

However we cannot simply integrate this equation, as we could before, to reduce the order of the ODE. Instead we multiply (4.14) by  $y$ , so that

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) y + \frac{2}{(2k+1)+2n} f y + \frac{1}{(2k+1)+2n} f' y^2 = 0,$$

and now it is possible to integrate by parts, to obtain

$$(-1)^{k+1} D_y^{2k}(|f|^n f) y + (-1)^k D_y^{2k-1}(|f|^n f) + \frac{1}{(2k+1)+2n} f y^2 = 0. \quad (4.15)$$

We also look at conservation of the second moment,

$$\int_{\mathbb{R}} x^2 u(x, t) \, dx = t^{3\beta-\alpha} \int_{\mathbb{R}} y^2 f(y) \, dy.$$

This gives

$$\alpha = \frac{3}{(2k+1)+3n},$$

so

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \frac{3}{(2k+1)+3n} f + \frac{1}{(2k+1)+3n} f' y = 0. \quad (4.16)$$

Similarly, as before, we can multiply by  $y^2$  and integrate to reduce the order, to obtain the ODE

$$\begin{aligned} & (-1)^{k+1} D_y^{2k}(|f|^n f) y^2 + 2(-1)^k D_y^{2k-1}(|f|^n f) y \\ & + 2(-1)^{k+1} D_y^{2k-2}(|f|^n f) + \frac{1}{(2k+1)+3n} f y^3 = 0. \end{aligned} \quad (4.17)$$

These three conservation laws in particular are important, as we can explicitly find the first three (second-order) equations for the case  $k = 1$  (corresponding to the first three nonlinear eigenvalues), but not for other  $k$ . The case  $k = 1$  is important, since it is much easier to develop theory for the lower order case, as well as it being easier to solve numerically (see Section 4.2.2).

In general, for  $l < 2k + 1$ , where  $l$  is the eigenvalue index as before, we have our moments conservation given by

$$\int_{\mathbb{R}} x^l u(x, t) \, dx = t^{(l+1)\beta-\alpha} \int_{\mathbb{R}} y^l f(y) \, dy.$$

Therefore our nonlinear eigenvalues may be represented by

$$\alpha_l(n) = \frac{l+1}{(2k+1)+(l+1)n}, \quad 0 \leq l < 2k + 1.$$

Our generalised NDE, representing all eigenvalues, is now reduced to the ODE

$$(-1)^{k+1} D_y^{2k+1}(|f|^n f) + \frac{l+1}{(2k+1)+(l+1)n} f + \frac{1}{(2k+1)+(l+1)n} f' y = 0. \quad (4.18)$$

However for  $l \geq 2k + 1$ , we cannot find  $\alpha_l$  explicitly using conservation laws, as these do not exist.



### 4.2.2 Numerical Construction of Nonlinear Eigenfunctions

In order to find reliable profiles for rescaled solutions of the NDE (4.7), a shooting method is used. First, considering solutions in which there is conservation of mass, we have our first “nonlinear eigenvalue” (for  $l = 0$ ), where  $n = 0$  corresponds to the linear kernel. Here the rescaled equation is given by (4.13). We assume that for small solutions of  $Y(y)$ , with  $y_0 < 0$ , we can approximate it by

$$Y(y) = C_0(y - y_0)_+^{\tilde{\alpha}}(1 + o(1)), \quad (4.19)$$

for some constant  $C_0$  and power  $\tilde{\alpha}$ . Here the following notation

$$(\cdot)_+ = \max\{0, \cdot\},$$

has been used. The lower order case  $k = 1$ , yields a second order equation, given by

$$Y'' = -\frac{1}{n+3} |Y|^{-\frac{n}{n+1}} Y y. \quad (4.20)$$

Substituting (4.19) into the ODE (4.20), we obtain the equation

$$\tilde{\alpha}(\tilde{\alpha} - 1)C_0(y - y_0)^{\tilde{\alpha}-2} = -\frac{1}{n+3} C_0^{\frac{1}{n+1}}(y - y_0)^{\frac{\tilde{\alpha}}{n+1}} y_0, \quad (4.21)$$

where we use  $(y - y_0)$  to mean  $(y - y_0)_+(1 + o(1))$ , as defined before. Hence we must have from (4.21), that

$$\tilde{\alpha} = \frac{2(n+1)}{n},$$

with  $2 < \tilde{\alpha} \leq 4$ , for  $n \geq 1$ . From this, the constant  $C_0$  is given by

$$C_0 = \left( \frac{n^2|y_0|}{2(n+1)(n+2)(n+3)} \right)^{\frac{n+1}{n}} \quad \text{for } y_0 < 0.$$

Hence small solutions can then be approximated by

$$Y = \left( \frac{n^2|y_0|}{2(n+1)(n+2)(n+3)} \right)^{\frac{n+1}{n}} (y - y_0)^{\frac{2(n+1)}{n}} (1 + o(1)),$$

with the derivative expansion

$$Y' = \frac{2(n+1)}{n} \left( \frac{n^2|y_0|}{2(n+1)(n+2)(n+3)} \right)^{\frac{n+1}{n}} (y - y_0)^{\frac{n+2}{n}} (1 + o(1)). \quad (4.22)$$

The Matlab Initial Value Problem solver *ode15s* is used, to plot profiles for these rescaled solutions. Taking an arbitrary initial point  $y = y_0 < 0$ , our solution and first derivative is expected to be zero here. Since both initial conditions are zero we will often find the solution  $Y = 0$ , whilst trying to solve numerically. In order to overcome this problem, we must look at some point  $y_0 + \delta$  (for small  $\delta$ ), close to this point. After finding the derivative there, this is used as the initial condition.

Obviously due to the nature of (4.22), we must take a relatively large initial point (in our case we take  $y_0 = -10$ ), in order for us to have an initial condition that is not negligible. Hence the solution is a large rescaling of any fundamental solution, which is also due to the scaling (4.12). Below are a few profiles that have been found, in which the value  $\delta = 10^{-3}$  has been taken. For  $n \lesssim 0.5$  the derivative  $Y'$  is very small and larger negative initial points must be used, to find reliable profiles. This makes comparison, between different values of  $n$ , more difficult.

In general, for all values of  $l$ , the lower order case of  $k = 1$  yields small solutions

$$Y = \left( \frac{n^2 |y_0|}{2(n+1)(n+2)((l+1)n+3)} \right)^{\frac{n+1}{n}} (y - y_0)^{\frac{2(n+1)}{n}} (1 + o(1)),$$

with derivative

$$Y' = \frac{2(n+1)}{n} \left( \frac{n^2 |y_0|}{2(n+1)(n+2)((l+1)n+3)} \right)^{\frac{n+1}{n}} (y - y_0)^{\frac{n+2}{n}} (1 + o(1)). \quad (4.23)$$

Here  $l < 2k + 1$  and hence we can only have  $l = 0, 1, 2$ .

When  $l = 1$ , we have from (4.15), with  $Y(y) = |f|^n f$  and  $k = 1$ , that

$$Y'' = \frac{1}{y} Y' - \frac{1}{2n+3} |Y|^{-\frac{n}{n+1}} Y y.$$

For  $l = 2$  and  $k = 1$ , we have from (4.17) that

$$Y'' = \frac{1}{y} 2Y' - \frac{1}{y^2} 2Y - \frac{1}{3n+3} |Y|^{-\frac{n}{n+1}} Y y.$$

These approximations are used to plot the profiles of our NDE with  $l = 1$  and  $l = 2$ . These correspond to  $F'$  and  $F''$  respectively, for the linear kernel  $F(y)$ , when  $n = 0$ .

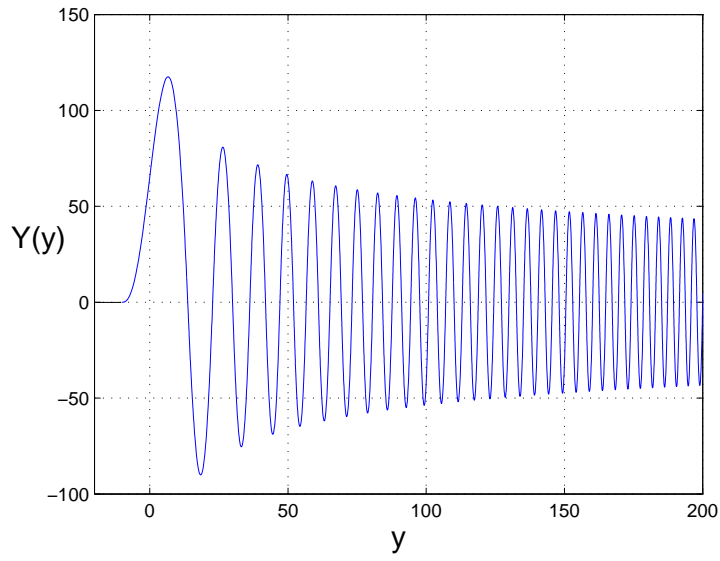


Figure 4-1: Rescaled solution  $Y(y)$ , for  $k = 1$  and  $l = 0$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , with  $n = 3$ .

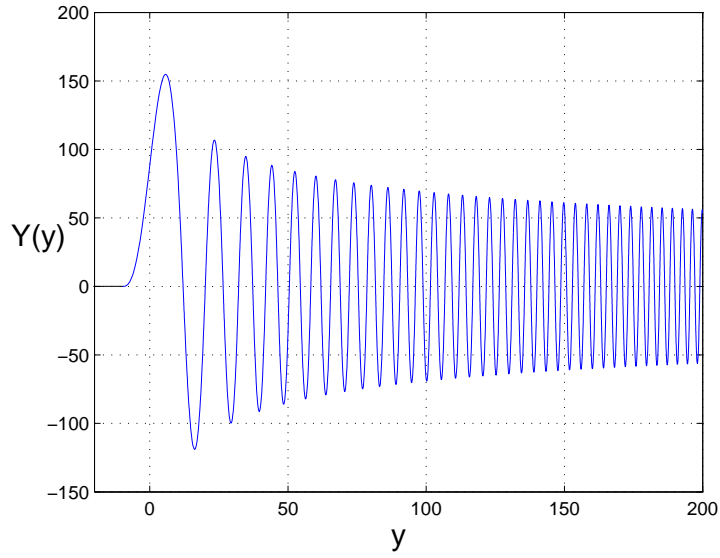


Figure 4-2: Rescaled solution  $Y(y)$ , for  $k = 1$  and  $l = 0$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , with  $n = 2$ .

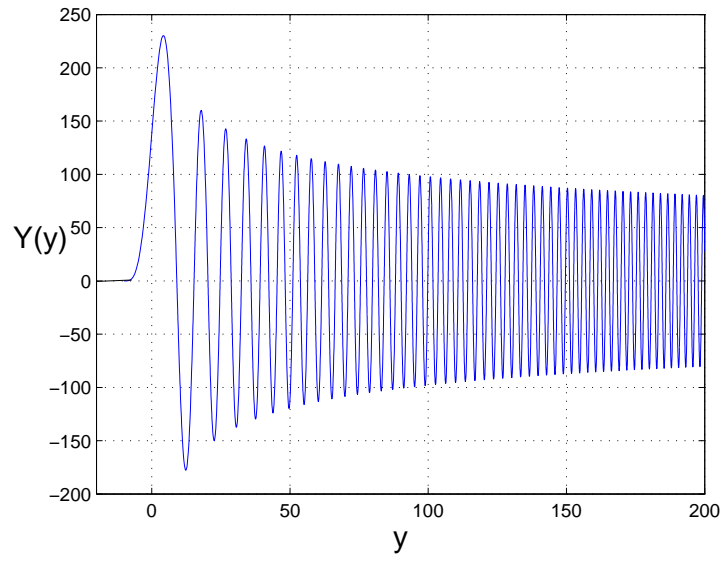


Figure 4-3: Rescaled solution  $Y(y)$ , for  $k = 1$  and  $l = 0$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , with  $n = 1$ .

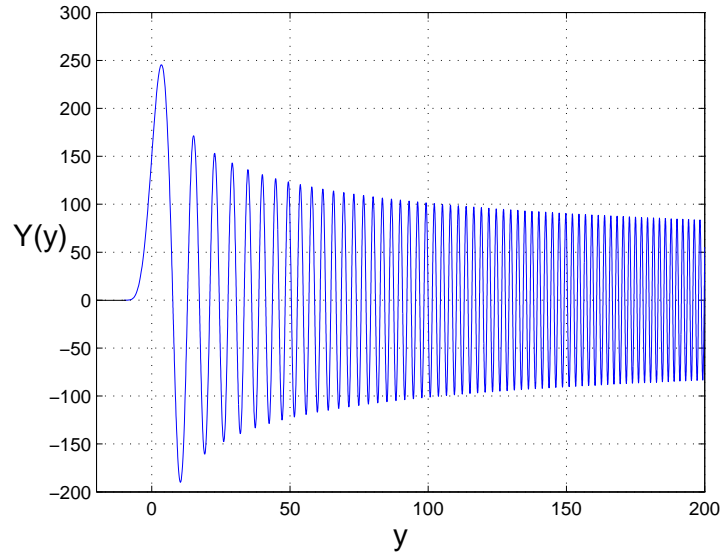


Figure 4-4: Rescaled solution  $Y(y)$ , for  $k = 1$  and  $l = 0$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , with  $n = 0.7$ .

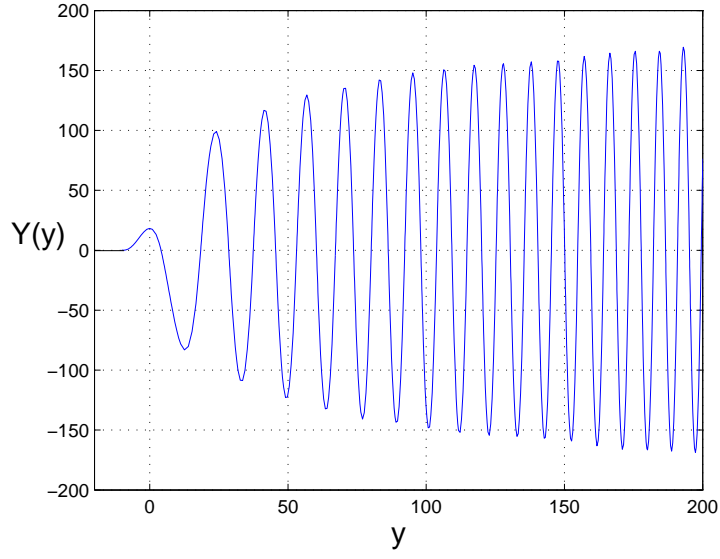


Figure 4-5: Rescaled solution  $Y(y)$ , for  $k = 1$  and  $l = 1$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , with  $n = 3$ .

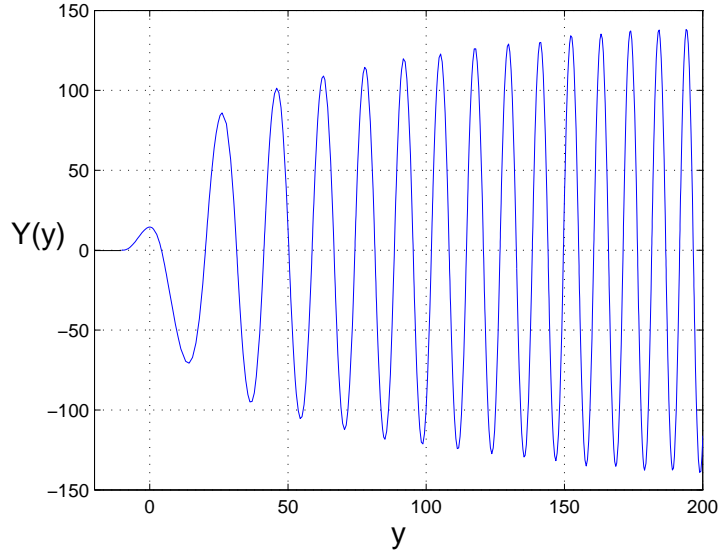


Figure 4-6: Rescaled solution  $Y(y)$ , for  $k = 1$  and  $l = 1$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , with  $n = 4$ .

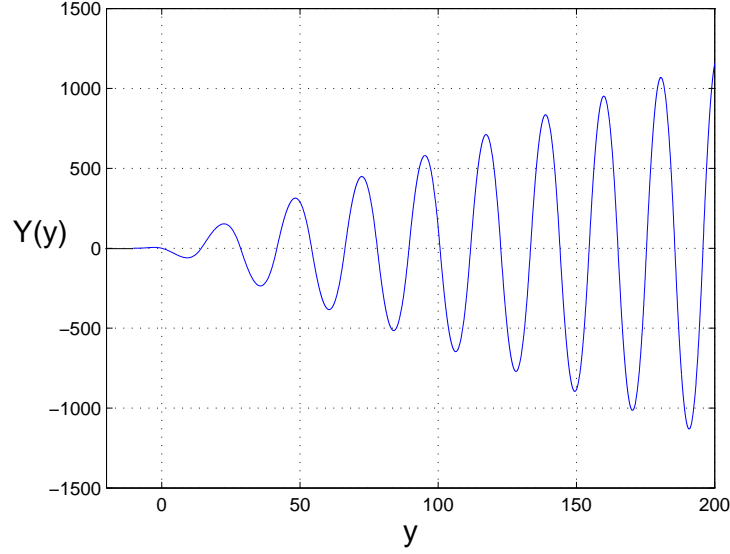


Figure 4-7: Rescaled solution  $Y(y)$ , for  $k = 1$  and  $l = 2$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , with  $n = 3$ .

### 4.3 Banach Contraction Principle: Local Existence and Uniqueness

In order to show that the above numerical construction can be justified, it is necessary to prove local existence and uniqueness, by showing that we can find a fixed point for the corresponding nonlinear integral equation. We consider our integrated second order equation, where  $k = 1$ ,

$$Y'' = -\frac{1}{n+3} Y^{\frac{1}{n+1}} y. \quad (4.24)$$

We rewrite our derivatives of  $Y$  in terms of  $y$

$$Y' = \frac{dY}{dy} = \frac{1}{y'(Y)}$$

and

$$Y'' = \frac{d}{dy} \left( \frac{1}{y'} \right) = -\frac{y''}{(y')^3}.$$

So now we can write our equation (4.24) in terms of  $y(Y)$  such that

$$\begin{aligned}
& -\frac{y''}{(y')^3} = -\frac{1}{n+3} Y^{\frac{1}{n+1}} y \\
\iff & -\frac{1}{2(y')^2} = -\frac{1}{n+3} \int_0^r y(s) s^{\frac{1}{n+1}} ds \\
\iff & y(Y) = y_0 + \int_0^Y \sqrt{\frac{n+3}{2 \int_0^r y(s) s^{\frac{1}{n+1}} ds}} dr \equiv M(y).
\end{aligned} \tag{4.25}$$

**Proposition 4.3.1**  *$M(y)$  is a contraction in  $C([0, \delta])$  and therefore admits a fix point.*

**Proof.** We need to show that for  $M(y)$  to be a contraction, then

$$\|M(\zeta_2) - M(\zeta_1)\| < \delta \|\zeta_2 - \zeta_1\|, \tag{4.26}$$

for some  $\delta \in (0, 1)$ .

It is easy to see that

$$M : Z_\delta \rightarrow Z_\delta,$$

for the space  $Z_\delta$  of continuous spaces, given by

$$Z_\delta = \{\zeta(Y) \in C([0, \delta])\}.$$

The norm is the supremum norm

$$\|\zeta\| := \sup_{Y \in [0, \delta]} |\zeta(Y)|.$$

Now let  $\zeta_1(Y), \zeta_2(Y) \in Z$ . From (4.25), we have that

$$\|M(\zeta_2) - M(\zeta_1)\| = \sqrt{\frac{n+3}{2}} \int_0^Y \left\| \left( \int \zeta_2(s) s^{\frac{1}{n+1}} ds \right)^{-\frac{1}{2}} - \left( \int \zeta_1(s) s^{\frac{1}{n+1}} ds \right)^{-\frac{1}{2}} \right\| dr,$$

where we use the simplified notation for the integral  $\int$ , without any limits of integration, to mean  $\int_0^r$ . This equation can now be written as

$$\|M(\zeta_2) - M(\zeta_1)\| = \sqrt{\frac{n+3}{2}} \int_0^Y \left\| \frac{\left( \int \zeta_2(s) s^{\frac{1}{n+1}} ds \right)^{-1} - \left( \int \zeta_1(s) s^{\frac{1}{n+1}} ds \right)^{-1}}{\left( \int \zeta_2(s) s^{\frac{1}{n+1}} ds \right)^{-\frac{1}{2}} + \left( \int \zeta_1(s) s^{\frac{1}{n+1}} ds \right)^{-\frac{1}{2}}} \right\| dr.$$

Using the substitution  $v = \frac{1}{n+1}$ , we see that

$$\begin{aligned} \|M(\zeta_2) - M(\zeta_1)\| &= \sqrt{\frac{n+3}{2}} \\ &\times \int_0^Y \left\| \frac{\int \zeta_1(s) s^v ds - \int \zeta_2(s) s^v ds}{(\int \zeta_2(s) s^v ds)(\int \zeta_1(s) s^v ds) \left( (\int \zeta_2(s) s^v ds)^{-\frac{1}{2}} + (\int \zeta_1(s) s^v ds)^{-\frac{1}{2}} \right)} \right\| dr. \end{aligned}$$

However, since we are looking at small values of  $y$ , then  $\zeta(s)$  is negligible and

$$\begin{aligned} \|M(\zeta_2) - M(\zeta_1)\| &\leq \mu_0 \int_0^Y \left\| \frac{\int (\zeta_1 - \zeta_2) s^v ds}{\int s^v ds \int s^v ds \left( (\int s^v ds)^{-\frac{1}{2}} + (\int s^v ds)^{-\frac{1}{2}} \right)} \right\| dr \\ &\leq \mu_0 \|\zeta_2 - \zeta_1\| \int_0^Y \frac{r^{\frac{n+2}{n+1}}}{r^{\frac{n+2}{n+1}} r^{\frac{n+2}{n+1}} r^{-\frac{n+2}{2(n+1)}}} dr \\ &\leq \mu_0 \|\zeta_2 - \zeta_1\| \int_0^Y r^{\frac{n+2}{2(n+1)}} dr \\ &\leq \mu_0 \|\zeta_2 - \zeta_1\| Y^{\frac{3n+4}{2(n+1)}}. \end{aligned}$$

Here,  $\mu_0$  is a constant dependent on  $n$  and  $y_0$ . Since we take  $\frac{1}{2}|y_0| \leq y \leq |y_0|$ , then we have that  $|Y| < 1$ , and so fixing  $Y \in [0, Y_0]$ , with  $\delta = \mu_0 |Y_0| < 1$ , we have that (4.26) holds true. Hence by *Banach's Fixed Point Theorem* [15, p. 39],  $M(y)$  has a unique fixed point, in  $Z_\delta$ .

□

## 4.4 Local and Global Behaviour

At this moment in time we do not know the behaviour of solutions for  $y \gg 1$ . Since solutions converge to our linear kernel  $F$ , as  $n \rightarrow 0^+$ , we also expect to have reasonably similar behaviour and so expect oscillations with decay as  $y \rightarrow +\infty$ , given by (2.7). However this is only true for when  $\alpha = \frac{1}{(2k+1)+n}$ , in (4.10), with  $\alpha$  being the first “nonlinear eigenvalue”. For other values of  $\alpha$ , we have behaviour similar to  $D^\beta F(y)$ , given by (2.36). Due to the complicated nature of the nonlinear equations, we are currently unable to find any sharp asymptotics and so look to find properties of the solutions using other methods.

**Proposition 4.4.1** *The solution  $f(y)$ , for (4.15), does not increase by power and so does not exhibit large unbounded monotone behaviour, as  $y \rightarrow +\infty$ .*



**Proof.** Let us first assume that  $f(y)$  does indeed have power behaviour, such that

$$f = Ay^m(1 + o(1)) \quad \text{as } y \rightarrow +\infty, \quad (4.27)$$

for some constant  $A$  and power  $m$ . Let us look at the case where  $k = 1$ , for our second eigenvalue  $\alpha_1$ . Substituting (4.27) into (4.15) yields for the leading term

$$\begin{aligned} m(n+1)[m(n+1)-1]A^{n+1}y^{m(n+1)-1} \\ -m(n+1)A^{n+1}y^{m(n+1)-1} + \frac{1}{3+2n}Ay^{m+2} = 0. \end{aligned} \quad (4.28)$$

By equating powers of  $y$ , we find the exponent  $m$  by

$$m(n+1)-1 = m+2 \implies m = \frac{3}{n}.$$

Now substituting  $m$  back into (4.28) we find that

$$A^n = -\frac{1}{(3+2n)(3+\frac{3}{n})(1+\frac{3}{n})}.$$

Since  $A^n < 0$ , it follows that there does not exist such  $A \in \mathbb{R}$ , such that  $f = Ay^m$ . Hence there are no large solutions with exponential behaviour.

Similarly, looking at the equation for the second moment (4.17), with  $f = Ay^m$ , we find that in this case

$$\begin{aligned} m(n+1)[m(n+1)-1]A^{n+1}y^{m(n+1)} - 2m(n+1)A^{n+1}y^{m(n+1)} \\ + 2A^{n+1} + \frac{1}{3+3n}Ay^{m+3} = 0. \end{aligned}$$

So our exponent is given by

$$m = \frac{3}{n}.$$

We then see that our constant  $A$  can be represented by

$$A^n = -\frac{1}{(3+2n)[9(1+\frac{1}{n})(\frac{1}{n})+2]}.$$

Therefore, as in the previous case,  $f \neq Ay^m$  and the solution  $f$  does not exhibit large behaviour.

Assume now, for  $k = 1$  in (4.13), that

$$Y \rightarrow +\infty \quad \text{monotonically as } y \rightarrow +\infty.$$

Then we see that

$$Y'' \leq -y.$$

Integrating this we find that

$$Y \leq -\frac{1}{6} C_0 y^3 + C_1 y. \quad (4.29)$$

However, since  $y \rightarrow +\infty$ , we see from (4.29), that  $Y \rightarrow -\infty$ , which is a contradiction. □

It is expected, and can be shown, that similar behaviour follows, for  $k = 2, 3, 4, \dots$  and for all other nonlinear eigenvalues  $\alpha_l(n)$ .

#### 4.4.1 A Priori Bounds for $Y(y)$ : Nonlinear Oscillatory Tail

Assuming solutions exist, we look to find any properties which define the behaviour, given that we now know  $f \neq Ay^m$ , for all  $A, m \in \mathbb{R}$ . If we can show that  $Y(y_1) > Y(y_2)$ , for  $y_1 < y_2$ , such that  $Y'(y_1) = Y'(y_2) = 0$ , then it will prove that  $Y(y)$  is a decreasing bounded function, as  $y \rightarrow +\infty$ .

We initially look at the equation for the first nonlinear eigenvalue,  $\alpha_0$ , for  $k = 1$ . Multiplying (4.20) by  $Y'$  and integrating between  $y_1, y_2 > 0$ , it can be shown that

$$\int_{y_1}^{y_2} Y'' Y' = -\frac{1}{n+3} \int_{y_1}^{y_2} |Y|^{-\frac{n}{n+1}} Y Y' y.$$

Simplifying this we see

$$\frac{1}{2} [(Y')^2]_{y_1}^{y_2} = -\frac{(n+1)}{(n+2)(n+3)} \int_{y_1}^{y_2} (|Y|^{\frac{n+2}{n+1}})' y.$$

After integrating by parts, the above equation reduces to

$$\frac{1}{2} [(Y')^2]_{y_1}^{y_2} = -\frac{(n+1)}{(n+2)(n+3)} [|Y|^{\frac{n+2}{n+1}} y]_{y_1}^{y_2} + \frac{(n+1)}{(n+2)(n+3)} \int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}}.$$

However, as was stated, we are looking at  $Y'(y_1) = Y'(y_2) = 0$ , hence

$$[(Y')^2]_{y_1}^{y_2} = 0,$$

and it follows that

$$|Y(y_2)|^{\frac{n+2}{n+1}} y_2 - |Y(y_1)|^{\frac{n+2}{n+1}} y_1 = \int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} dy.$$

Since the following identity

$$\int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} dy > 0,$$

is true, then

$$|Y(y_2)|^{\frac{n+2}{n+1}} y_2 > |Y(y_1)|^{\frac{n+2}{n+1}} y_1.$$

Or rearranging we see

$$\left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{\frac{n+2}{n+1}} > \frac{y_1}{y_2}.$$

If  $y_2 > y_1$ , then this means that  $Y(y_2)$  cannot be small in comparison with  $Y(y_1)$ .

We continue by looking at our second nonlinear eigenvalue  $\alpha_1$ , where our equation is given by (4.15). Multiplying by  $Y'$  once again, we obtain

$$\begin{aligned} Y'Y''y - (Y')^2 + \frac{1}{3+2n}|Y|^{-\frac{n}{n+1}}YY'y^2 &= 0 \\ \implies \frac{1}{2}[(Y')^2]'y - (Y')^2 + \frac{n+1}{(n+2)(3+2n)}(|Y|^{\frac{n+2}{n+1}})'y^2 &= 0. \end{aligned}$$

Integrating between  $y_1$  and  $y_2$  again, we have that

$$\begin{aligned} \frac{1}{2}[(Y')^2y]_{y_1}^{y_2} - \frac{3}{2} \int_{y_1}^{y_2} (Y')^2 + \frac{n+1}{(n+2)(3+2n)} [|Y|^{\frac{n+2}{n+1}}y^2]_{y_1}^{y_2} \\ - \frac{n+1}{(n+2)(3+2n)} \int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} 2y = 0 \\ \implies -\frac{3}{2} \int_{y_1}^{y_2} (Y')^2 - \frac{n+1}{(n+2)(3+2n)} \int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} 2y \\ + \frac{n+1}{(n+2)(3+2n)} (|Y(y_2)|^{\frac{n+2}{n+1}} y_2^2 - |Y(y_1)|^{\frac{n+2}{n+1}} y_1^2) = 0. \end{aligned}$$

One can see that

$$\frac{3}{2} \int_{y_1}^{y_2} (Y')^2 + \frac{n+1}{(n+2)(3+2n)} \int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} 2y > 0,$$

since  $y > 0$ . Hence we have that

$$|Y(y_2)|^{\frac{n+2}{n+1}} y_2^2 > |Y(y_1)|^{\frac{n+2}{n+1}} y_1^2,$$

or

$$\left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{\frac{n+2}{n+1}} > \left( \frac{y_1}{y_2} \right)^2.$$

Finally for the third eigenvalue  $\alpha_2$ , for (4.17), multiplying by  $Y'$  yields

$$\begin{aligned} Y'Y''y^2 - 2(Y')^2y + 2YY' + \frac{1}{3+3n}Y^{-\frac{n}{n+1}}YY'y^3 &= 0 \\ \implies \frac{1}{2}[(Y')^2]'y^2 - 2(Y')^2y + (Y^2)' + \frac{n+1}{(n+2)(3+3n)}(|Y|^{\frac{n+2}{n+1}})'y^3 &= 0. \end{aligned}$$

Integrating between  $y_1$  and  $y_2$

$$\begin{aligned} \frac{1}{2}[(Y')^2y^2]_{y_1}^{y_2} - 3 \int_{y_1}^{y_2} (Y')^2y + [Y^2]_{y_1}^{y_2} + \frac{n+1}{(n+2)(3+3n)}[|Y|^{\frac{n+2}{n+1}}y^3]_{y_1}^{y_2} \\ - \frac{n+1}{(n+2)(3+3n)} \int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} 3y^2 = 0. \end{aligned}$$

So

$$\left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{\frac{n+2}{n+1}} > \left( \frac{y_1}{y_2} \right)^3.$$

In general it can easily be seen that

$$\left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{\frac{n+2}{n+1}} > \left( \frac{y_1}{y_2} \right)^{l+1}.$$

These estimates characterise the behaviour of the nonlinear oscillatory tail, as seen in the numerics in Section 4.2.2. Unfortunately due to the nature of the method used, estimates for  $k \geq 2$ , cannot be found this way.

#### 4.4.2 Oscillatory Structure and Periodicity

As mentioned before, we expect a behaviour which is structurally similar to the linear kernel and its derivatives. We therefore expect to have oscillatory behaviour for  $y \gg 1$ . Hence we look to describe this oscillatory structure and try to find if these oscillations are given by periodic functions. Let us introduce the oscillatory component  $\phi(s)$ , where we have that

$$Y(y) = y^\gamma \phi(s) \quad \text{for } s = \ln y, \tag{4.30}$$

for some power  $\gamma \in \mathbb{R}$ . Here the term  $y^\gamma$  gives the rate of any growth/decay of the oscillations and may be compared to the controlling factor  $y^{-\frac{2k-1}{4k}}$ , found in the linear asymptotics. For  $k = 1$ , substituting into (4.18), we see that

$$(y^\gamma \phi)''' + \frac{l+1}{3+(l+1)n} y^{\frac{\gamma}{n+1}} \phi^{\frac{1}{n+1}} + \frac{1}{3+(l+1)n} (y^{\frac{\gamma}{n+1}} \phi^{\frac{1}{n+1}})' y = 0.$$

Expanding this expression and equating powers of  $y$ , we find that

$$\gamma = \frac{3(n+1)}{n}.$$

This gives us the third order nonlinear differential equation

$$\begin{aligned} \phi''' + \frac{3(2n+3)}{n} \phi'' + \frac{11n^2+36n+27}{n^2} \phi' + \frac{3(n+1)(2n+3)(n+3)}{n^3} \phi \\ + \frac{1}{n} \phi^{\frac{1}{n+1}} + \frac{1}{3(n+1)+(l+1)(n+1)n} \phi^{-\frac{n}{n+1}} \phi' = 0. \end{aligned} \quad (4.31)$$

However we note that

$$\frac{3(n+1)}{n} \rightarrow \infty \quad \text{as} \quad n \rightarrow 0,$$

so conclude oscillatory behaviour (4.30), does not have the necessary structure and is not applicable. Hence we can also say that any oscillatory solutions are not given by periodic oscillatory components  $\phi(\ln y)$ .

In the general case, for all  $k$ , we find that

$$(-1)^k D_y^{2k+1}(y^\gamma \phi) = \frac{l+1}{(2k+1)+(l+1)n} y^{\frac{\gamma}{n+1}} \phi^{\frac{1}{n+1}} + \frac{1}{(2k+1)+(l+1)n} (y^{\frac{\gamma}{n+1}} \phi^{\frac{1}{n+1}})',$$

for (4.30). This gives a  $2k+1$  order nonlinear differential equation of the form

$$(-1)^k P_{2k+1}(\phi) = \frac{1}{n} \phi^{\frac{1}{n+1}} + \frac{1}{(2k+1)(n+1)+(l+1)(n+1)n} \phi^{-\frac{n}{n+1}} \phi'.$$

Here,  $P_{2k+1}(\phi)$  is a polynomial operator on  $\phi$ , given by the term

$$D_y^{2k+1}(y^\gamma \phi).$$

For the case  $k = 1$ , the polynomial operator was given by

$$P_3(\phi) = \phi''' + \frac{3(2n+3)}{n} \phi'' + \frac{9n^2+7n+27}{n} \phi' + \frac{3(n+1)(2n+3)(n+3)}{n^3} \phi.$$

In general,  $P_{2k+1}(\phi)$  can be given by the recursion

$$P_{2k+1}(\phi) = \frac{d^2}{ds^2} P_{2k-1}(\phi(s)) + (\gamma - 2k) \frac{d}{ds} P_{2k-1}(\phi(s)) + (\gamma - 2k + 1) \frac{d}{ds} P_{2k-1}(\phi(s)) \\ + (\gamma - 2k)(\gamma - 2k + 1) P_{2k-1}(\phi(s)).$$

For all  $k \geq 0$ , it can easily be seen that

$$\gamma = \frac{(2k+1)(n+1)}{n}.$$

Hence, as before, oscillatory solutions of the form (4.30), for any  $k$ , are not applicable.

## 4.5 Branching

We now apply another classic idea, to trace out the behaviour of all the nonlinear eigenfunctions, for small  $n > 0$ . Namely it shall be shown that there is branching of solutions, with respect to the parameter  $n$ . However, we do not intend to get any rigorous mathematical proof of branching. Let us look at the general NDE given by (4.10). We first expand  $|f|^n$  to formally get

$$|f|^n = 1 + n \ln |f| + o(n).$$

This is precisely true in any positivity set  $\{|f| \geq \delta_0 > 0\}$ . However, note that we do not at this moment discuss any rigorous functional settings of this expansion, for  $y \in \mathbb{R}$ . Then (4.10) reduces to

$$(-1)^{k+1} D^{2k+1} [(1 + n \ln |f|)f] + \alpha f + \frac{1-\alpha n}{2k+1} f' y + o(n) = 0.$$

Expanding coefficients for small  $n > 0$ , yields

$$(\mathbf{B} - \lambda_l I) f + (-1)^{k+1} D^{2k+1} (n \ln |f| f) + \left(\alpha - \frac{l+1}{2k+1}\right) f - \frac{\alpha n}{2k+1} f' y + O(n^2) = 0, \quad (4.32)$$

where  $\lambda_l = -\frac{l}{2k+1}$ .

For  $l < 2k + 1$ , we can find our eigenvalues  $\alpha_l$  explicitly and represent it as

$$\alpha_l = \frac{l+1}{(2k+1)+n(l+1)} = \frac{l+1}{2k+1} \left(1 + \frac{n(l+1)}{2k+1}\right)^{-1} = \frac{l+1}{2k+1} \left(1 - \frac{n(l+1)}{2k+1} + O(n^2)\right).$$

Then (4.32) reduces to

$$(\mathbf{B} - \lambda_l I)f + (-1)^{k+1} D^{2k+1}(n \ln |f|f) - \frac{n(l+1)^2}{(2k+1)^2} f - \frac{n(l+1)}{(2k+1)^2} f' y + O(n^2) = 0.$$

Hence using the Lyapunov-Schmidt reduction for

$$f = \psi_l + n\phi_l + O(n^2), \quad (4.33)$$

we have that

$$(\mathbf{B} - \lambda_l I)\phi_l = (-1)^k D^{2k+1}(\ln |\psi_l| \psi_l) + \frac{(l+1)^2}{(2k+1)^2} \psi_l + \frac{(l+1)}{(2k+1)^2} \psi'_l y + O(n^2) \equiv h.$$

It now remains to show that the right-hand side is orthogonal to  $\psi_l^*$ , by showing that  $\langle h, \psi_l^* \rangle_* = 0$ . This is known as the orthonormality condition, in the classic Lyapunov-Schmidt method.

We use our definition of our adjoint polynomial eigenfunctions  $\psi_l^*$ , given by (2.34). Then for  $l < 2k + 1$  we have that

$$\psi_l^* = \frac{1}{\sqrt{l!}} y^l$$

So

$$\begin{aligned} \langle h, \psi_l^* \rangle_* &= \frac{1}{\sqrt{l!}} \int \left[ (-1)^k D^{2k+1}(\ln |\psi_l| \psi_l) y^l + (-1)^l \frac{(l+1)^2}{(2k+1)^2} \psi_l y^l \right. \\ &\quad \left. + (-1)^l \frac{(l+1)}{(2k+1)^2} \psi'_l y^{l+1} \right] dy \\ &= \frac{1}{\sqrt{l!}} \int \left[ (-1)^k D^{2k+1}(\ln |\psi_l| \psi_l) y^l + (-1)^l \frac{(l+1)}{(2k+1)^2} (\psi_l y^{l+1})' \right] dy = 0, \end{aligned}$$

for  $l < 2k + 1$ .

However, for  $l \geq 2k + 1$  we do not know the values of  $\alpha_l$  explicitly. In this case, we set

$$\alpha_l(n) = \alpha_0 + n\alpha_1 + O(n^2). \quad (4.34)$$

As before, we use (4.32) and now we substitute (4.34) as well as (4.33) to obtain

$$\begin{aligned} n(\mathbf{B} - \lambda_l I)\phi_l &= (-1)^k D^{2k+1}(n \ln |\psi_l| \psi_l) + \left( \alpha_0 + n\alpha_1 - \frac{l+1}{2k+1} \right) \psi_l \\ &\quad + n \left( \alpha_0 - \frac{l+1}{2k+1} \right) \phi_l - \frac{n\alpha_0}{2k+1} \psi'_l y + O(n^2) = 0. \end{aligned}$$

Equating coefficients of  $O(n)$  we can find the value of  $\alpha$ , with

$$\alpha_0 \psi_l - \frac{l+1}{2k+1} \psi_l = 0,$$

and

$$\begin{aligned} n(\mathbf{B} - \lambda_l I) \phi_l &= (-1)^k D^{2k+1} (n \ln |\psi_l| \psi_l) + n \alpha_1 \psi_l \\ &\quad + n \left( \alpha_0 - \frac{l+1}{2k+1} \right) \phi_l - \frac{n \alpha_0}{2k+1} \psi'_l y. \end{aligned} \quad (4.35)$$

Hence

$$\alpha_0 = \frac{l+1}{2k+1},$$

and substituting into (4.35)

$$(\mathbf{B} - \lambda_l I) \phi_l = (-1)^k D^{2k+1} (\ln |\psi_l| \psi_l) + \alpha_1 \psi_l - \frac{l+1}{(2k+1)^2} \psi'_l y \equiv h.$$

Taking inner product with  $\psi_l^*$ , noting that we must have  $\langle h, \psi_l^* \rangle_* = 0$  and  $\langle \psi_l, \psi_l^* \rangle_* = 1$ , we obtain

$$\alpha_l = \langle (-1)^k D^{2k+1} (\ln |\psi_l| \psi_l) - \frac{l+1}{(2k+1)^2} \psi'_l y, \psi_l^* \rangle_*,$$

which means we have  $\alpha_l(n)$  for all values of  $l$ .

So the above shows that we have all formal mathematical aspects of branching of solutions, with respect to the parameter  $n \rightarrow 0$ . This shows a “homotopic path”, from nonlinear eigenfunctions to linear ones. Proving such branching phenomena is expected to be extremely mathematically difficult, and this is not a part of the current research.

## 4.6 Limiting Behaviour as $n \rightarrow +\infty$ : Highly Non-linear Case

We know that as  $n \rightarrow 0^+$ , there exists certain convergence to solutions of our linear PDE (2.1), but now we also look to find the behaviour of our NDEs as  $n \rightarrow +\infty$ . Consider the NDE (4.13), with  $k = 1$  and where  $l = 0$ . Since we are finding the limiting behaviour as  $n \rightarrow +\infty$ , it is necessary to scale out any



coefficients containing  $n$ . In order to do this, we let

$$Y = C\tilde{Y},$$

for some constant  $C(n)$ . Then substituting into (4.13) yields

$$C\tilde{Y}'' = -\frac{1}{n+3}|C|^{-\frac{n}{n+1}}C|\tilde{Y}|^{-\frac{n}{n+1}}\tilde{Y}y.$$

So now we have

$$|C| = (n+3)^{-\frac{n+1}{n}},$$

and scaling out  $C$  we obtain the ODE

$$\tilde{Y}'' = -|\tilde{Y}|^{-\frac{n}{n+1}}\tilde{Y}y.$$

Hence, as  $n \rightarrow +\infty$ , we have that  $-\frac{n}{n+1} \rightarrow -1$ , so for  $n = \infty$ , the ODE takes the simpler form

$$\begin{aligned}\tilde{Y}'' &= -\frac{\tilde{Y}}{|\tilde{Y}|}y \\ &= -\operatorname{sgn}(\tilde{Y})y.\end{aligned}\tag{4.36}$$

Solving this ODE, we find two separate equations dependent on the sign of  $\tilde{Y}$ :

$$\begin{cases} \tilde{Y} > 0 : & \tilde{Y}_+ = -\frac{1}{6}y^3 + c_1y + c_2, \\ \tilde{Y} < 0 : & \tilde{Y}_- = \frac{1}{6}y^3 + d_1y + d_2. \end{cases}\tag{4.37}$$

Here  $c_1, c_2, d_1, d_2$  are all constants, though not necessarily positive ones.

Knowing the conditions of continuity for the function  $\tilde{Y}(y)$ , we must have that  $\tilde{Y}_+ = \tilde{Y}_-$  and  $\tilde{Y}'_+ = \tilde{Y}'_-$ , when  $\tilde{Y} = 0$ .

Let the points  $y = y_i$  be such that,  $\tilde{Y}(y_i) = 0$ , for  $i = 0, 1, 2, \dots$ . Hence for  $\tilde{Y}_+(y_i)$ , given in (4.37),

$$\tilde{Y}_+(y_i) = -\frac{1}{6}y_i^3 + c_1y_i + c_2 = 0.$$

We now rearrange this to find one of the unknown parameters, in terms of  $y_i$  and the parameter  $c_2$ , such that

$$c_1 = \frac{1}{6}y_i^2 - \frac{c_2}{y_i}.$$

We also have that  $\tilde{Y}'_+(y_i) = \tilde{Y}'_-(y_i)$ , hence

$$-\frac{1}{2}y_i^2 + c_1 = \frac{1}{2}y_i^2 + d_1.$$

From this we find a second parameter in terms of  $y_i$  and  $c_2$ , where

$$d_1 = -\frac{5}{6}y_i^2 - \frac{c_2}{y_i}.$$

Now we see that from  $\tilde{Y}_+ = \tilde{Y}_-$ ,

$$c_1y_i + d_1y_i + c_2 + d_2 = 0.$$

So substituting in known values, we have our third parameter  $d_2$ , given by

$$d_2 = \frac{2}{3}y_i^3 + c_2.$$

After substituting values for  $c_1, d_1$  and  $d_2$ , (4.37) can be written as

$$\begin{cases} \tilde{Y} > 0 : & \tilde{Y}_+ = -\frac{1}{6}y^3 + (\frac{1}{6}y_i^2 - \frac{c_2}{y_i})y + c_2, \\ \tilde{Y} < 0 : & \tilde{Y}_- = \frac{1}{6}y^3 - (\frac{5}{6}y_i^2 + \frac{c_2}{y_i})y + \frac{2}{3}y_i^3 + c_2. \end{cases} \quad (4.38)$$

From the above, it is noted that  $y_i \neq 0$ , for any  $i$ , unless  $c_2 = 0$ .

Whilst despite being able to find a full solution dependent on just two parameters, we do not hope that this is an effective procedure in determining solutions and so we must find other ways to describe the behaviour of  $\tilde{Y}(y)$ . To solve the problem, it would be necessary to have one more condition to eliminate one of the free parameters. However (4.38) characterises the oscillatory behaviour of the rescaled solution.

For the general case of all higher order NDEs, where  $l = 0$ , the equations for  $n \rightarrow +\infty$  are much the same. Here the scaling constant  $C(n)$ , for  $Y = C(n)\tilde{Y}$ , can easily be shown to be given by

$$|C| = [(2k+1) + n]^{-\frac{n+1}{n}}. \quad (4.39)$$

After scaling, this yields the ODE

$$(-1)^{k+1}D_y^{2k}\tilde{Y} = -\text{sgn}(\tilde{Y})y.$$

### 4.6.1 Numerics as $n \rightarrow +\infty$

While we cannot explicitly determine the solution of our NDE, as  $n \rightarrow +\infty$ , even up to a single parameter, we can however easily find its behaviour numerically. Since the solution is given by the ODE (4.36), it is possible to just use a simple shooting method using the ODE solver ode45, to find profiles.

We use a similar shooting method as that used for the general case of  $n$ , set out in Section 4.2.2. We recall that we look for small solutions of  $Y(y)$ , with interface at  $y_0 < 0$ , such that

$$Y(y) = C_0(y - y_0)_+^{\tilde{\alpha}}(1 + o(1)),$$

where as before  $C_0$  is some constant and  $\tilde{\alpha}$  is at present some unknown power.

In the case of  $n \rightarrow +\infty$ , we find that on substitution into the ODE (4.36) and equating powers, that

$$\tilde{\alpha} = 2.$$

Hence the constant  $C_0$  is defined by

$$|C_0| = \frac{1}{2} |y_0|.$$

We then have that

$$\tilde{Y} = \frac{1}{2} |y_0| (y - y_0)^2 (1 + o(1)) \tag{4.40}$$

and

$$\tilde{Y}' = |y_0| (y - y_0) (1 + o(1)), \tag{4.41}$$

for small values of  $\tilde{Y}$ , as  $n \rightarrow +\infty$ . Once again we let  $y_0 = -10$  and plot the profiles.

Using the same method, the equations governing the behaviour as  $n \rightarrow +\infty$ , relating to the second and third nonlinear eigenvalues, can easily be found. For  $l = 1$  in the NDE (4.15), with  $k = 1$ , the equation as  $n \rightarrow +\infty$  can be given by the ODE

$$\tilde{Y}'' y - \tilde{Y}' + \frac{\tilde{Y}}{|\tilde{Y}|} y^2 = 0.$$

In general, for all values of  $k$ , we have

$$(-1)^{k+1} D_y^{2k} \tilde{Y} y + (-1)^k D_y^{2k-1} \tilde{Y} + \frac{\tilde{Y}}{|\tilde{Y}|} y^2 = 0,$$

as  $n \rightarrow +\infty$ . Similarly for the second eigenvalue, where  $l = 2$ , we have the lower order case  $k = 1$  given by

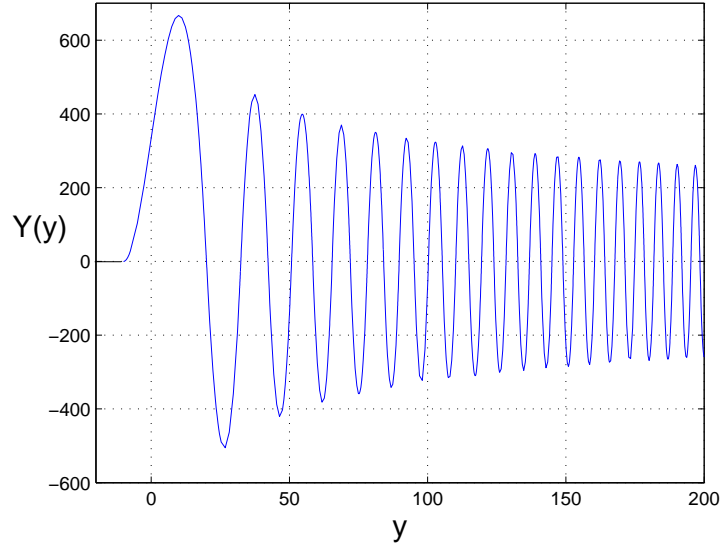
$$\tilde{Y}''y^2 - 2\tilde{Y}'y + 2\tilde{Y} + \frac{\tilde{Y}}{|\tilde{Y}|}y^3 = 0,$$

with general cases of  $k$  given by

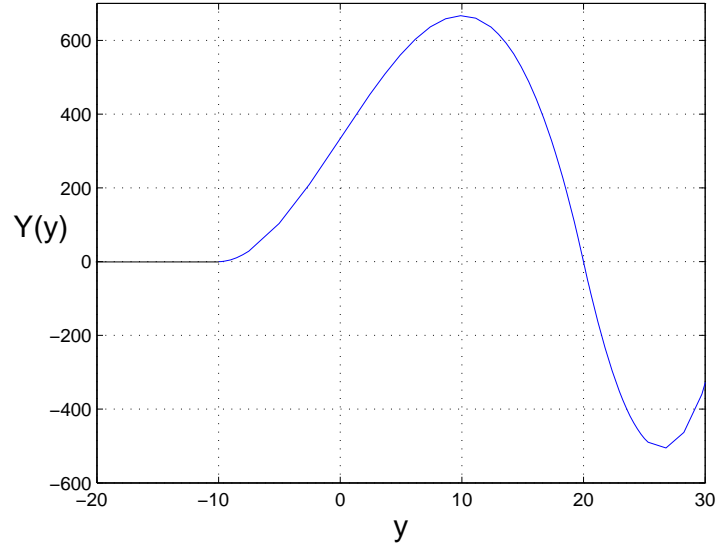
$$(-1)^{k+1}D_y^{2k}\tilde{Y}y^2 + 2(-1)^kD_y^{2k-1}\tilde{Y}y + 2(-1)^{k+1}D_y^{2k-2}\tilde{Y} + \frac{\tilde{Y}}{|\tilde{Y}|}y^3 = 0.$$

We note that for  $n \rightarrow +\infty$ , all higher eigenvalues have the same shooting parameters given by the fundamental eigenvalue in (4.40) and (4.41).

Naturally these solutions  $\tilde{Y}(y)$ , are a large rescaling of not only the original rescaled solution  $f(y)$ , but also  $Y(y)$ . Indeed taking a relatively large value for  $n$ , say  $n = 100$ , and plotting after scaling by (4.39), we can find that the behaviour is already very close to that of  $n = \infty$ .



(a) Global structure of the solution.



(b) Finite Interface.

Figure 4-8: Rescaled solution  $\tilde{Y}(y)$ , for  $k = 1$  and  $l = 0$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , as  $n \rightarrow +\infty$ .

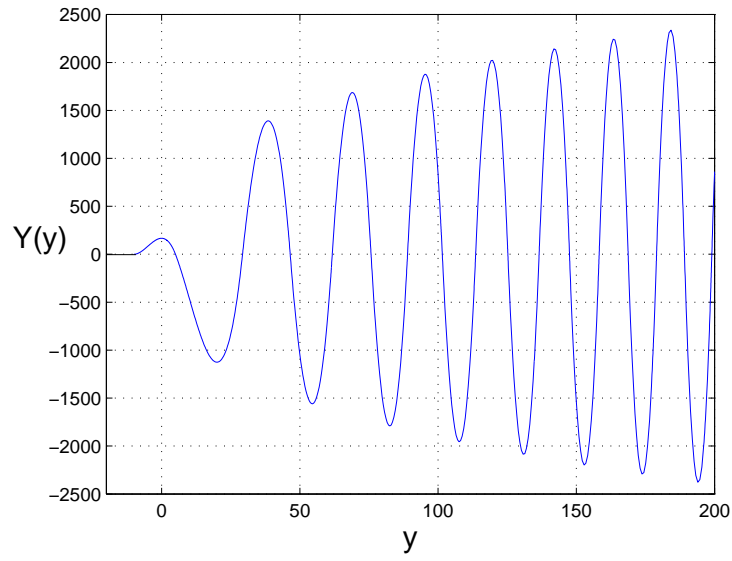


Figure 4-9: Rescaled solution  $\tilde{Y}(y)$ , for  $k = 1$  and  $l = 1$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , as  $n \rightarrow +\infty$ .

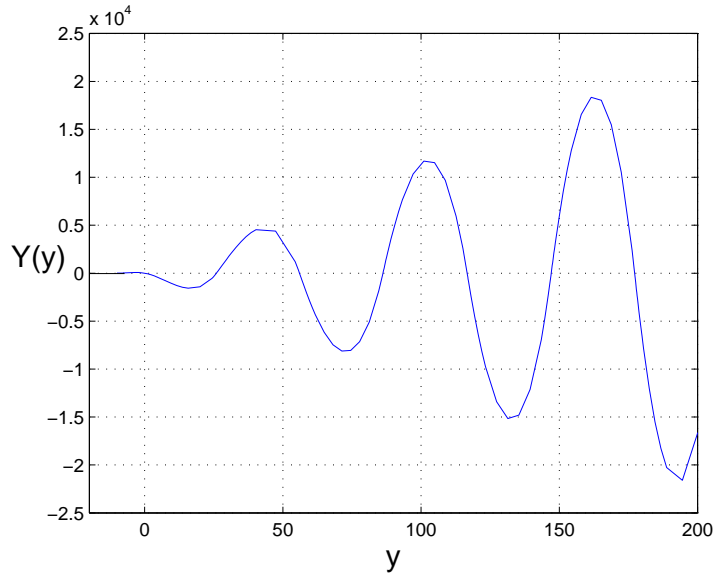


Figure 4-10: Rescaled solution  $\tilde{Y}(y)$ , for  $k = 1$  and  $l = 2$ , to the NDE  $u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u)$ , as  $n \rightarrow +\infty$ .

## 4.7 Nonlinear Dispersion with Absorption

The final natural progression, from the nonlinear model (4.7), would be to go to the odd-order NDE with absorption, given by

$$u_t = (-1)^{k+1} D_x^{2k+1} (|u|^n u) - |u|^{p-1} u \quad \text{in } \mathbb{R} \times \mathbb{R}_+. \quad (4.42)$$

This links up both the nonlinear model and the semilinear model (3.1). We do not however study this equation in detail and only briefly look at some aspects of the equation.

The PDE (4.42), after similarity scaling, reduces to the equation

$$(-1)^{k+1} D_y^{2k+1} (|f|^n f) + \frac{1}{p-1} f + \frac{(p-1)-n}{(p-1)(2k+1)} f' y - |f|^{p-1} f = 0, \quad (4.43)$$

for  $n > 0$ ,  $p > n + 1$ . We note that naturally, for the case  $n = 0$ , we have the semilinear model (3.1).

For the case  $n = 0$  and  $p = 1$ , in (4.42), we reduce to the equation

$$u_t = (-1)^{k+1} D_x^{2k+1} u - u.$$

We can see that by using the substitution

$$u(x, t) = e^{-t} v(x, t),$$

we reduce to a rescaled linear equation of the form

$$v_t = (-1)^{k+1} D_x^{2k+1} v.$$

Hence all linear theory formed in Chapter 2, may be used.

### 4.7.1 Numerical Construction

We briefly attempt to find numerical solutions to the equation (4.43). As usual, we look at the lower order case,  $k = 1$ . Once again, in order to remove the nonlinearity in the highest differential, the substitution

$$Y = |f|^n f,$$

is used. This yields the third order equation

$$Y''' + \frac{1}{p-1} |Y|^{-\frac{n}{n+1}} Y + \frac{(p-1)-n}{3(p-1)} (|Y|^{-\frac{n}{n+1}} Y)' y - |Y|^{\frac{p-n-1}{n+1}} Y = 0.$$

Due to the complexity of the equation, there is not much hope of solving this problem using a shooting method and hence we return to the BVP problem. However, even using this method, it is extremely difficult to produce any reliable numerics, due to the highly nonlinear nature of the problem, even for small values of  $p$  and  $n$ .

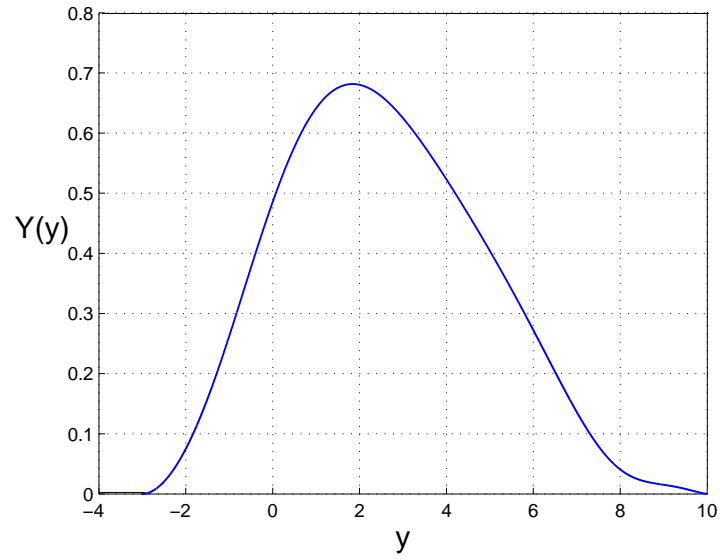
We must be careful when plotting, as we must avoid any nonlinear bifurcation points in  $p$ . In the figures below, we take  $p = 2.7$  and  $n = 0.5$ . For a small plot interval, we are able to obtain a stable profile. However, once we attempt to plot any oscillatory part, we cannot obtain any reliable convergence of solutions.

As before, with the semilinear case, there is difficulty in finding the correct boundary points, such that the correct oscillations are found. Figure 4.11(b), shows the attempts at finding the oscillatory part of the solution (where we take the second boundary point at  $y = 15$ ), though in this figure, the solution did not converge within the required tolerance. Figure 4.11(a) shows the main structure of the solution, which converges for a boundary point of  $y = 10$ .

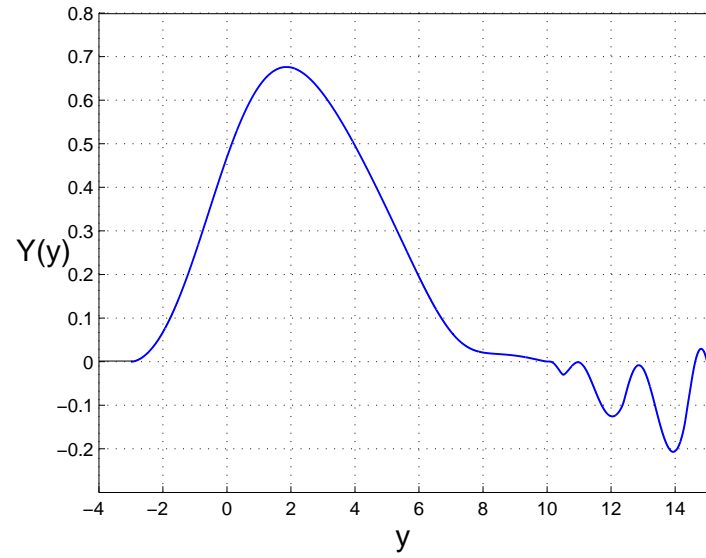
We do note that whilst the oscillatory part is very difficult to obtain, the non-oscillatory structure is very stable and does not change much, when changing the length in which the problem is evaluated on (see Figure 4-12).

As in Section 4.6, one way of finding the oscillatory behaviour, would be to look at the behaviour as  $n \rightarrow +\infty$  and reduce the ODE. However, in this case, this is much more difficult, due to the parameter  $p$ . Here we have that  $n$  is more difficult to scale from any coefficients in the equation and so this is not something we touch upon at this moment, though it is something that may be done in the future.





(a) Plot length  $y = (-3, 10)$ .



(b) Plot length  $(-3, 15)$ .

Figure 4-11: The rescaled solution,  $Y(y)$ , to  $u_t = (|u|^n u)_{xxx} - u^p$  with  $n = 0.5$  and  $p = 2.7$ .

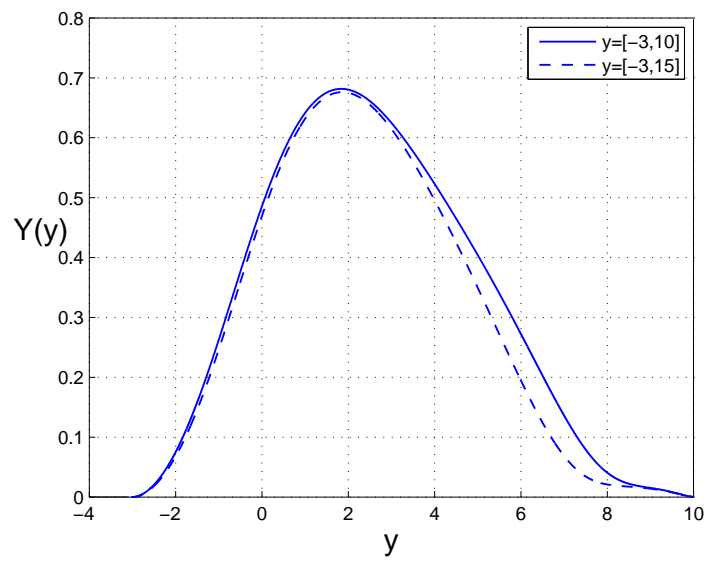


Figure 4-12: Comparison of rescaled solutions,  $Y(y)$ , to  $u_t = (|u|^n u)_{xxx} - u^p$ , with  $n = 0.5$  and  $p = 2.7$ .

# Chapter 5

## Conclusions and Further Work

### 5.1 Conclusions

This thesis has focused on finding behaviour for generalised odd-order evolutionary PDEs. We have looked at similarity solutions as a way of reducing the complexity of the problem and have attempted to gain knowledge in how these rescaled solutions behave. In particular many different techniques, which have been applied to related even-order problems, are used here. These even-order models are in turn based on the lower order case, the Heat equation and associated nonlinear PDEs (e.g. the Heat equation with absorption). These include asymptotic, analytical and numerical techniques, which all help to describe properties of the rescaled solution. However, problems with functional settings and dual space theory, are encountered, due to the highly oscillatory nature and this leads to several results to still remain formal.

In Chapter 2, we looked at the general linear problem, where it is known that the lower order case of  $k = 1$  yields the Airy equation. Whilst the behaviour, for all  $k$ , can be described by asymptotic analysis, which fully characterise the rescaled kernels, as  $y \rightarrow \pm\infty$ , spectral theory is needed in order to help describe behaviour for nonlinear models. It is known that the non self-adjoint linear operator  $\mathbf{B}$ , has point spectrum. Problems however occur with the duality between the operator  $\mathbf{B}$  and its adjoint  $\mathbf{B}^*$ . Indeed it can be seen that  $\mathbf{B}^*$  is not even adjoint in the standard metric of  $L^2(\mathbb{R})$ . Instead any dual products must be calculated in the indefinite metric of  $\bar{L}^2(\mathbb{R})$ . Formally, it can be shown that these dual products exist, as well as standard bi-orthonormality properties of the eigenfunctions and adjoint eigenfunctions.

In Chapter 3, the basis of the work done for the linear model is now applied to the semilinear absorption model. We looked for solutions of the rescaled equation, in the subcritical range of  $1 < p < p_0 = 2k + 2$ , where we assume  $p_0$  to be a bifurcation point. Numerically some reliable plots can be found for various values of  $p$ , in this range. But using numerics to prove that there is a bifurcation point at  $p = 4$ , for the lower order case of  $k = 1$ , has found to be difficult. We have found characteristics of the rescaled solutions, both in terms of their centre subspace behaviour and linearised stability.

The nonlinear model in Chapter 4, now has the nonlinearity in the highest differential. As in the case of related even-order nonlinear equations, reducing to an ODE using similarity solutions will not explicitly yield the full equation. In order to find the unknown parameter, conservation laws must be looked at. Indeed these conservation laws will show that this unknown parameter is equivalent to the nonlinear eigenvalues for the equation. We find that there is branching of solutions, with respect to the parameter  $n$ . Whilst the case  $n = 0$  will yield the linear equation, the most nonlinear case, as  $n \rightarrow +\infty$ , will yield a solvable differential, dependent on two unknown parameters. Some local and global properties, for all cases of  $n$  have been found, though full asymptotic behaviour has not.

## 5.2 Further Work

The achieved results are due to be published in two papers [18, 19], which are in preparation. The analysis of both linear and nonlinear models needs further extension and better understanding. Whilst there is a good understanding of the behaviour of the rescaled solutions, there are a few results which still remain formal.

Due to the fact that these models have not been studied much (if at all, in some cases), there is certainly much that can still be done, both in terms of models that have been looked at here and other related nonlinear odd-order models, which could be looked at in the future.

For the linear PDE, there are still some questions over functional settings and dual space theory, where some results are still formal. This in turn leaves some analysis in both the semilinear and nonlinear dispersion equations open, in particular with regards to bifurcation and branching. However, whilst the results are formal, they do give a good understanding of the behaviour.

In the case of the NDE, we would like to have found full asymptotic behaviour for the rescaled solutions. In particular analysis for higher order cases, for  $k > 1$ , has been difficult and more work on these could be looked. Not only are higher order cases still relatively unknown, but also certain aspects of higher “nonlinear eigenvalues”, which correspond to conservation of higher moments. Analysing these will lead to a better understanding of our highly oscillatory eigenfunctions,  $\psi_l(\beta)$ , for the LDEs.

In general, as hinted before, another direction could also possibly be to extend all these models into multi-dimensional spaces, where we would still retain some basic theory.

There are certainly a large a number of other odd-order models which could be investigated. Obviously at this stage any physical applications of these may not be known, but these models mainly follow on from related even-order parabolic models, that have already been studied.

One natural step from the semilinear odd-order model with absorption, would be to look at the odd-order model with source. Here the equation is given by

$$u_t = (-1)^{k+1} D_x^{2k+1} u + |u|^{p-1} u, \quad \text{in } \mathbb{R} \times \mathbb{R}_+. \quad (5.1)$$

It is known that the present VSS research on the semilinear PDE, with absorption, also embraces the well-known area of blow-up in solutions, for which is expected to occur in (5.1). Under the rescaling of variables, such that

$$t \mapsto T - t \quad \text{and} \quad x \mapsto -x,$$

it can be seen that equation (5.1) reduces to the semilinear equation with absorption, which we have studied. Hence we have also described classes of blow-up solutions.

Whilst the nonlinear dispersion equation with absorption has been briefly looked at, very little is known about it. Of course the absorption term could also be replaced here, with a source-like term. Another extension from the NDE could be to possibly look into odd-order thin-film type equations, where there would be numerous different models that could be looked at.

As stated several times before, this area of odd-order evolutionary PDEs is still relatively untouched and there is much that is not understood, plenty of scope for further analysis and new models. However, it is clear that a full study

of such nonlinear PDEs will need new mathematical tools.

# Appendix A

## Further Numerics for the Semilinear Model

Let us recall the odd-order semilinear dispersion model with absorption,

$$u_t = (-1)^{k+1} D_x^{2k+1} u - |u|^{p-1} u \quad \text{in } \mathbb{R} \times \mathbb{R}_+,$$

where we continue with some more numerical examples of these rescaled solutions.

We start by showing some examples of wrong profiles found, whilst solving the boundary value problem. It was explained in Section 3.2.1, that there are numerous difficulties that can be encountered when solving. Figures A-1 to A-4 show just a small number of profiles found, which are assumed to be inaccurate due to applying the wrong boundary point.

Figure A-1 is an example of not having oscillatory behaviour that is symmetric about  $f(y) = 0$ . It can be seen that a wrong choice of boundary point here, has produced artificial oscillatory behaviour, such that the rescaled solution is strictly positive.

At first glance it seems that Figure A-2, would be a reasonably accurate solution. However, it can be seen that the oscillations are not quite symmetric enough and that any symmetry does not occur close enough to  $y = 0$ . Whilst the case might not be as extreme as the previous example, there are still doubts over the reliability of this profile.

In Figure A-3, whilst there is symmetry of oscillatory behaviour, there is increasing behaviour in the solution. This is not to be expected, in light of the decay for linear solutions. Whilst in Figure A-4, we do not have minimisation of

the tail.

We also include some more profiles, which we assume to satisfy the conditions which we look for, to obtain the best possible profiles. Figures A-5 to A-15 show some further reliable profiles, for rescaled solutions of the semilinear dispersion model (3.1). Here the results take values such that

$$2 \leq p \leq 3.2.$$

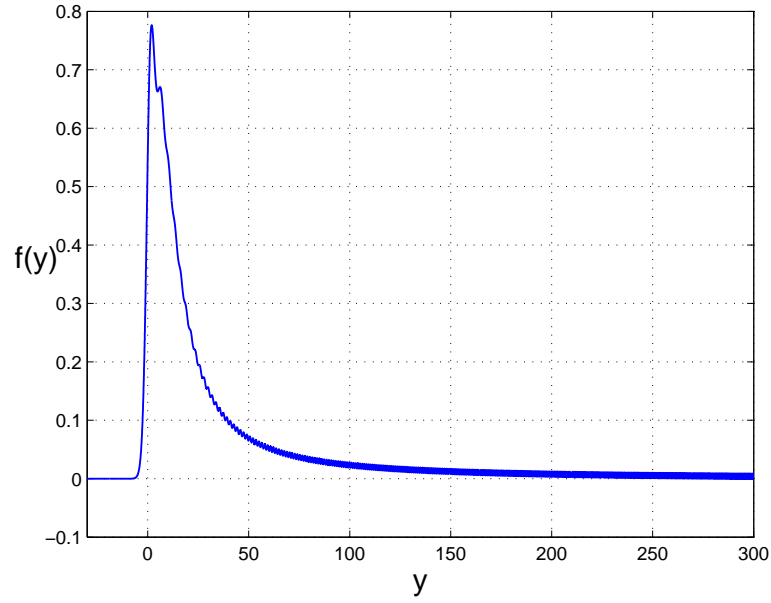
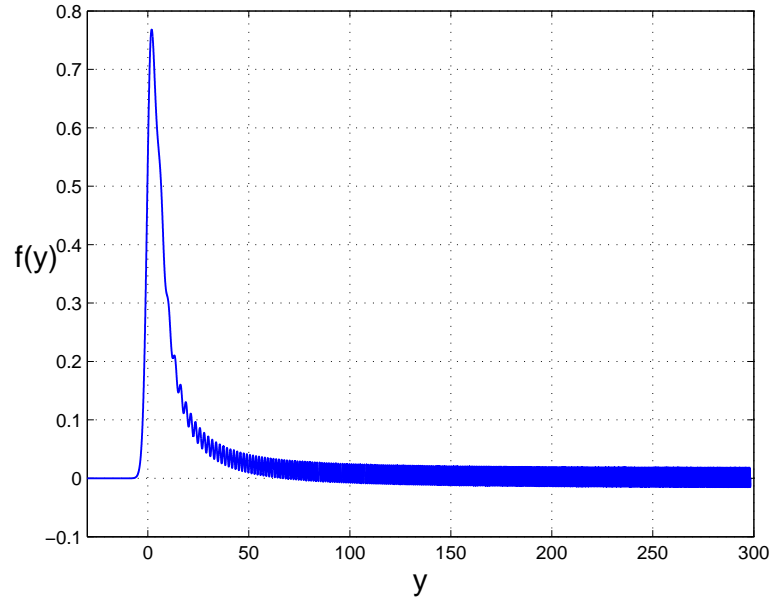
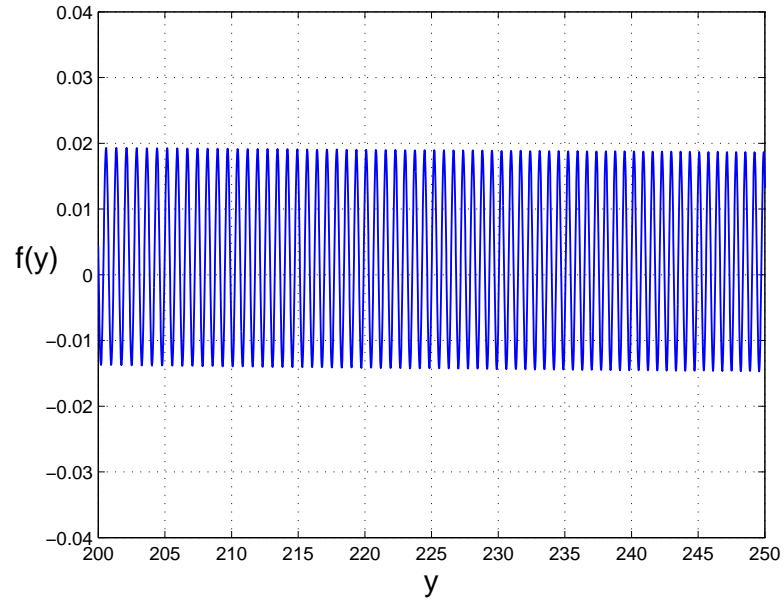


Figure A-1: Wrong VSS profile to  $u_t = u_{xxx} - u^p$ , with  $p = 2.9$ : oscillation above the axis.





(a) Global solution structure.



(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-2: Wrong VSS profile to  $u_t = u_{xxx} - u^p$ , with  $p = 2.9$ : tail is not sufficiently symmetric.

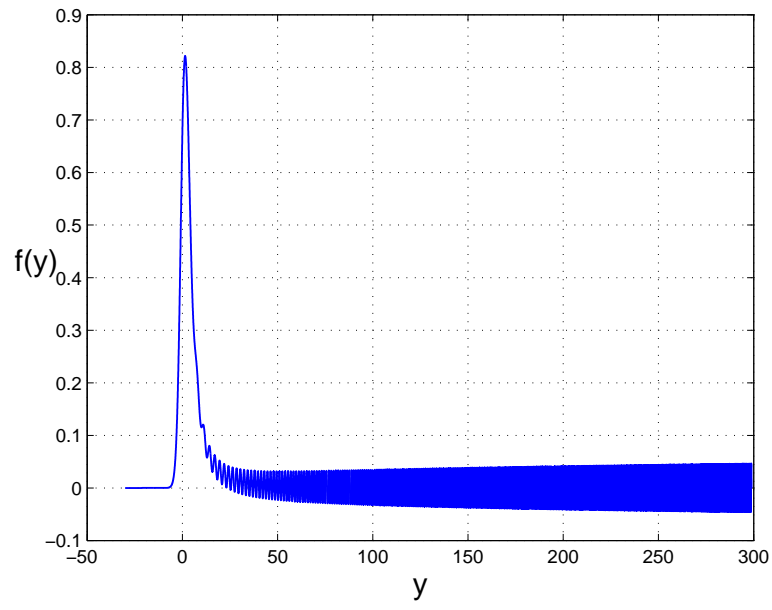


Figure A-3: Wrong profile to  $u_t = u_{xxx} - u^p$ , with  $p = 2.4$ : tail is increasing.

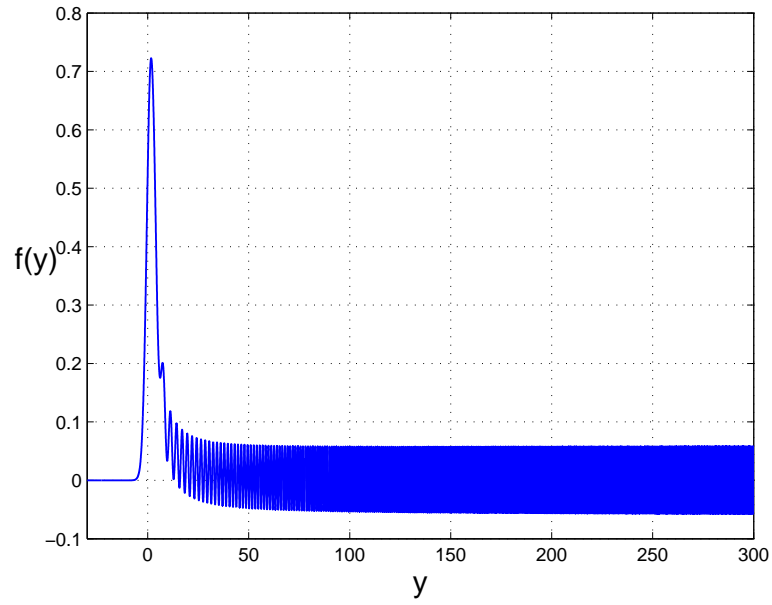
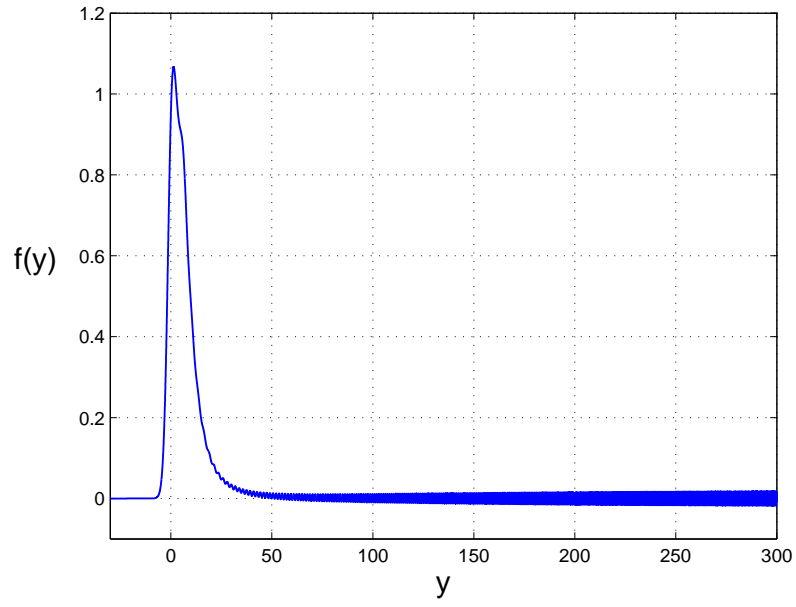
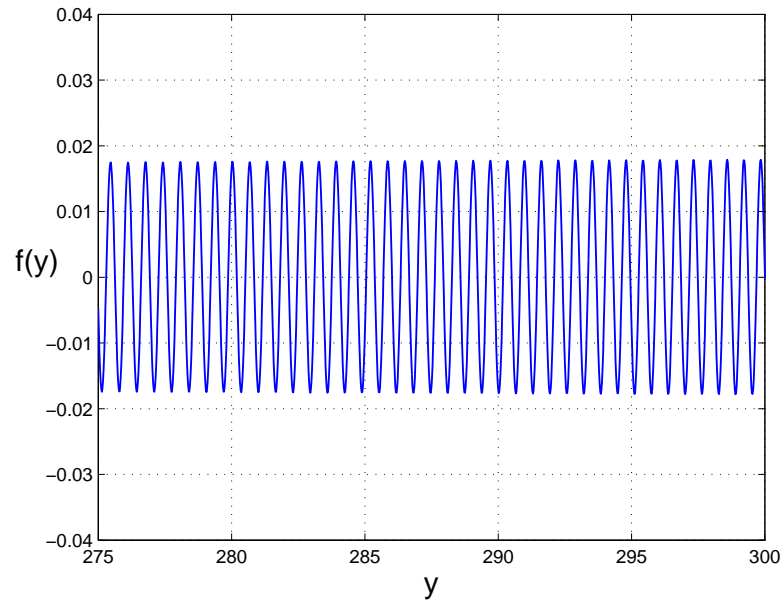


Figure A-4: Wrong VSS profile to  $u_t = u_{xxx} - u^p$ , with  $p = 2.9$ : tail is too large.

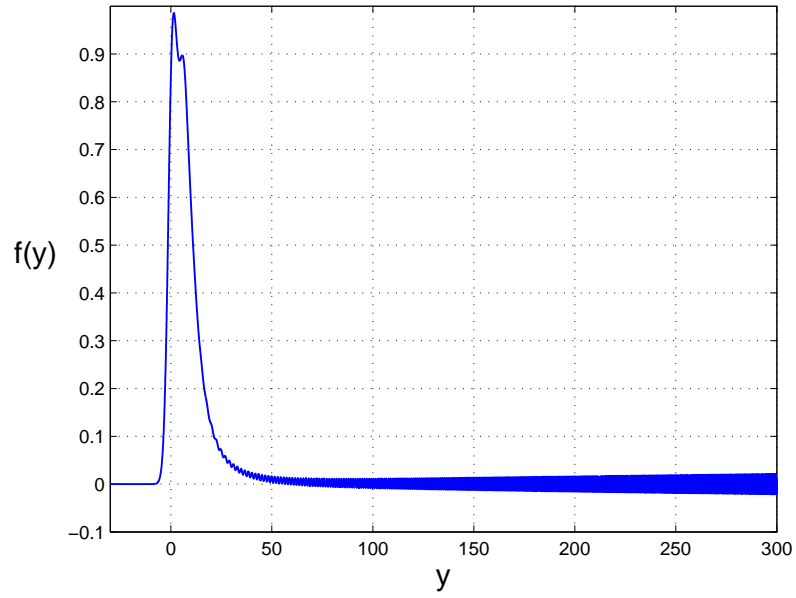


(a) Global solution structure.

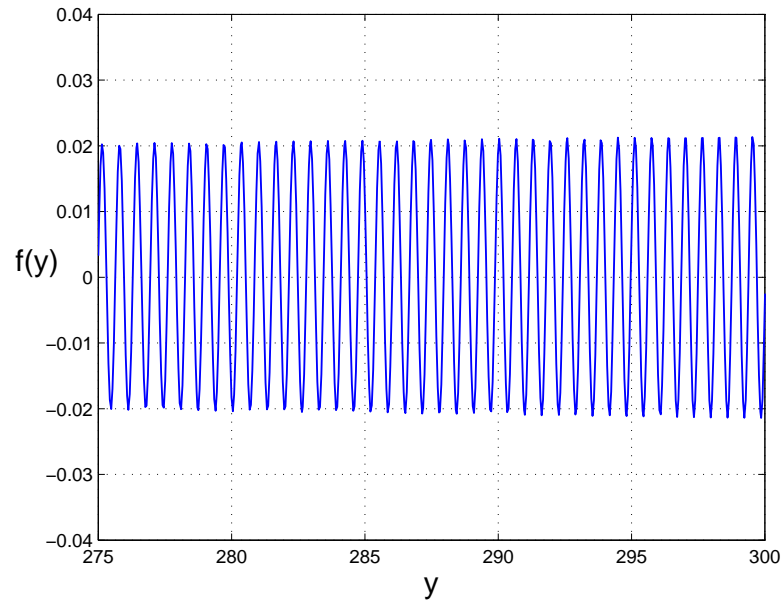


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-5: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2$ .

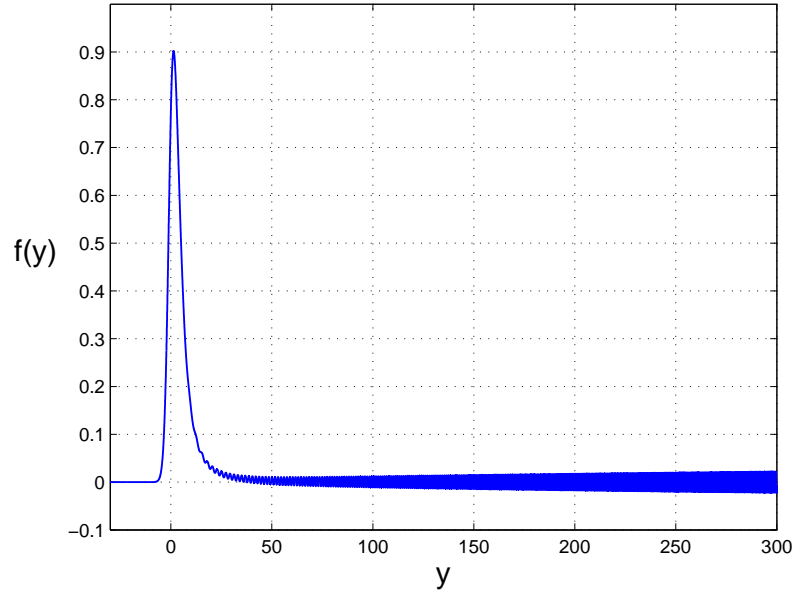


(a) Global solution structure.

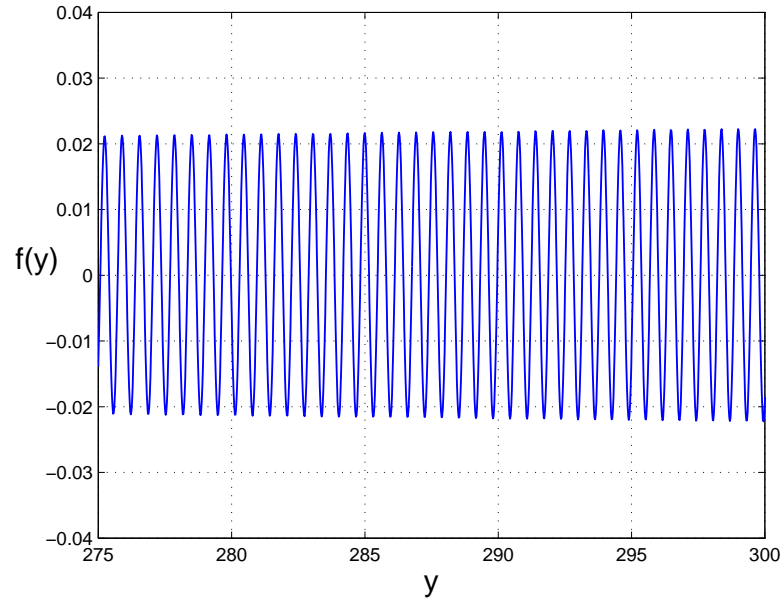


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-6: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.1$ .

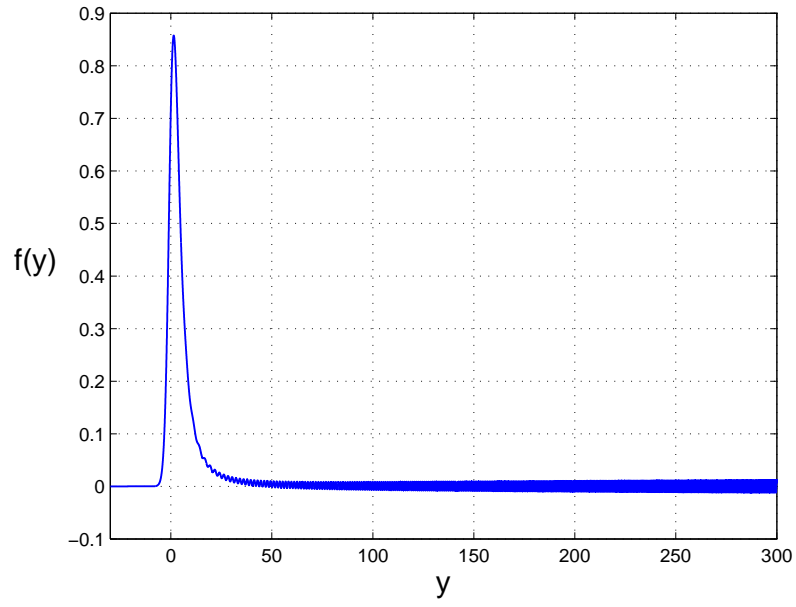


(a) Global solution structure.

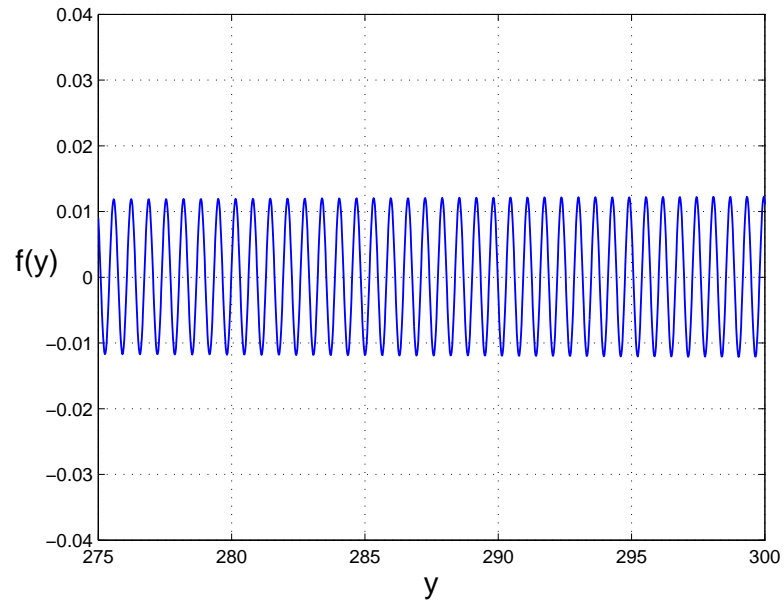


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-7: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.2$ .

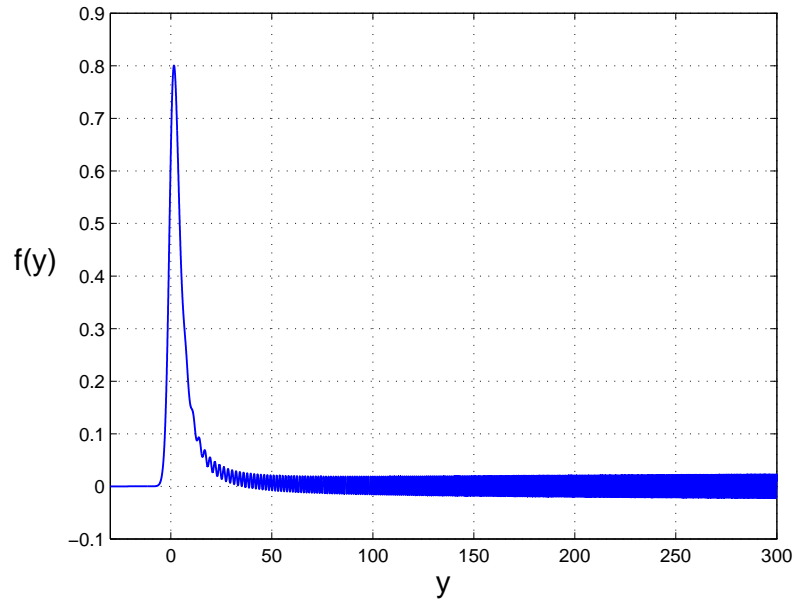


(a) Global solution structure.

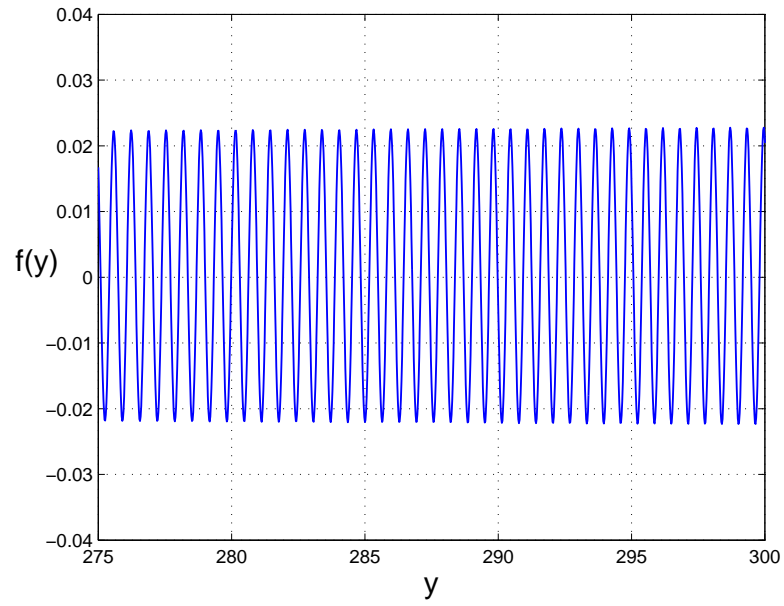


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-8: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.3$ .

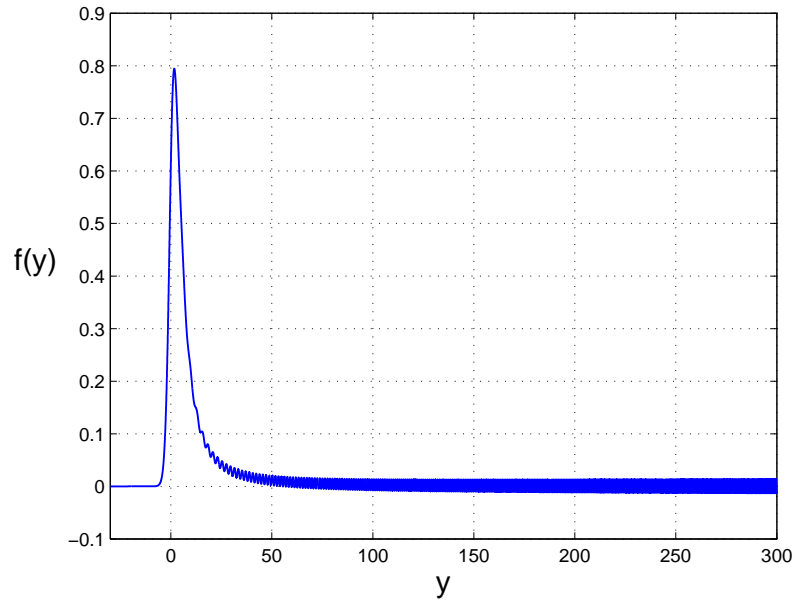


(a) Global solution structure.

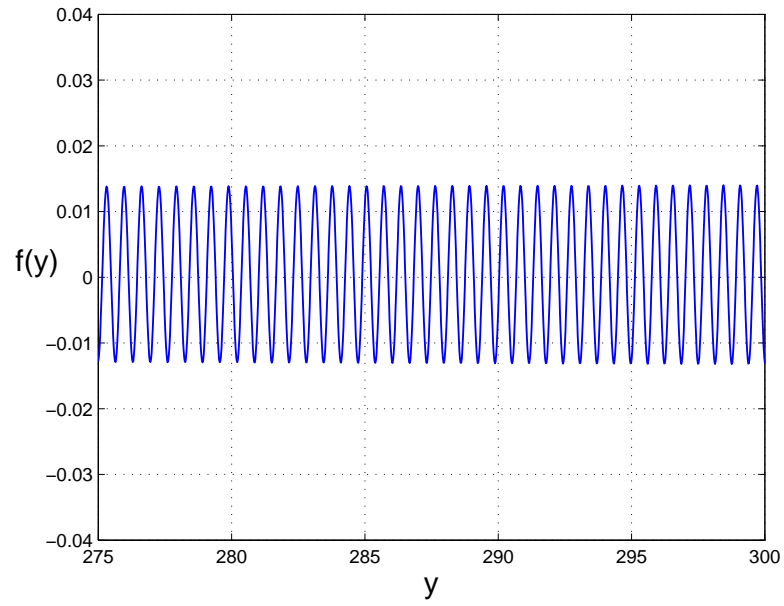


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-9: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.5$ .



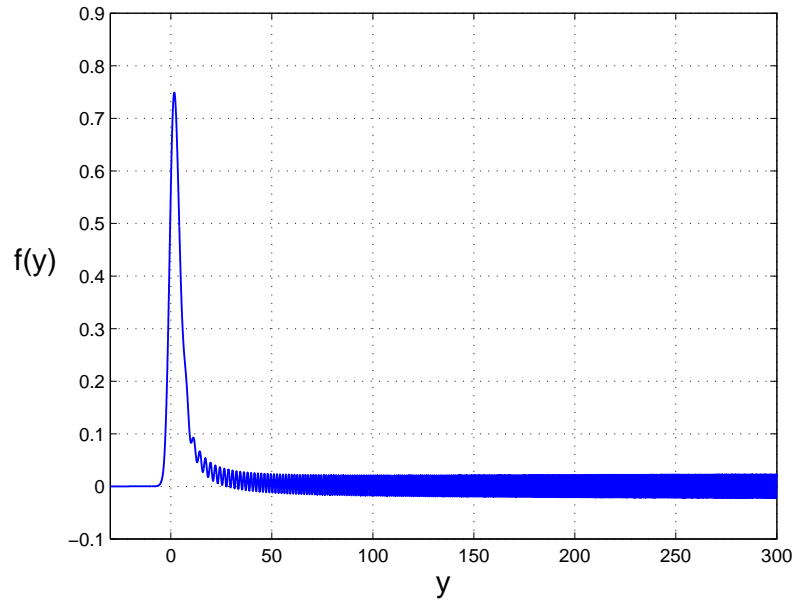
(a) Global solution structure.



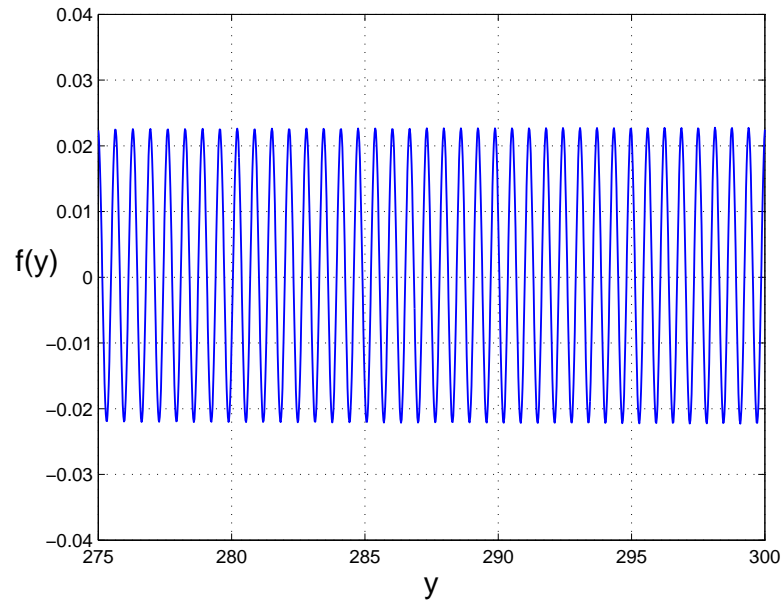
(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-10: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.6$ .



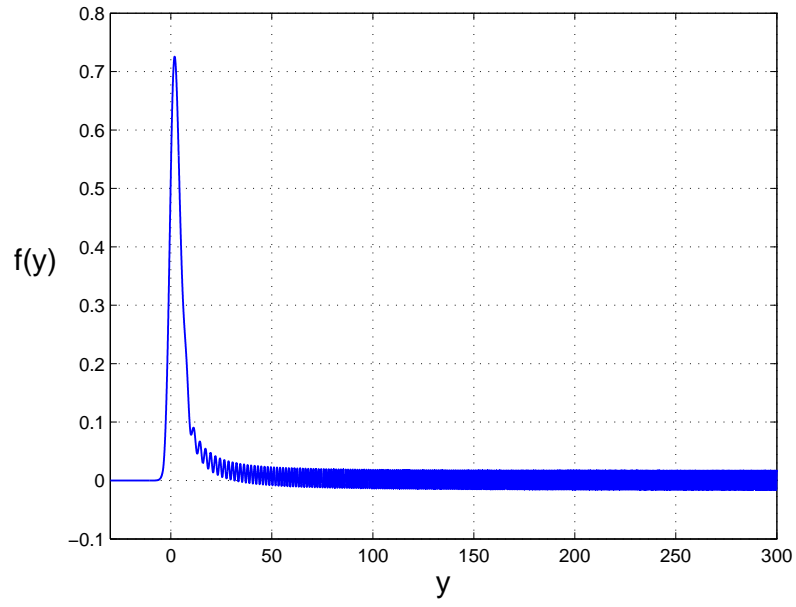


(a) Global solution structure.

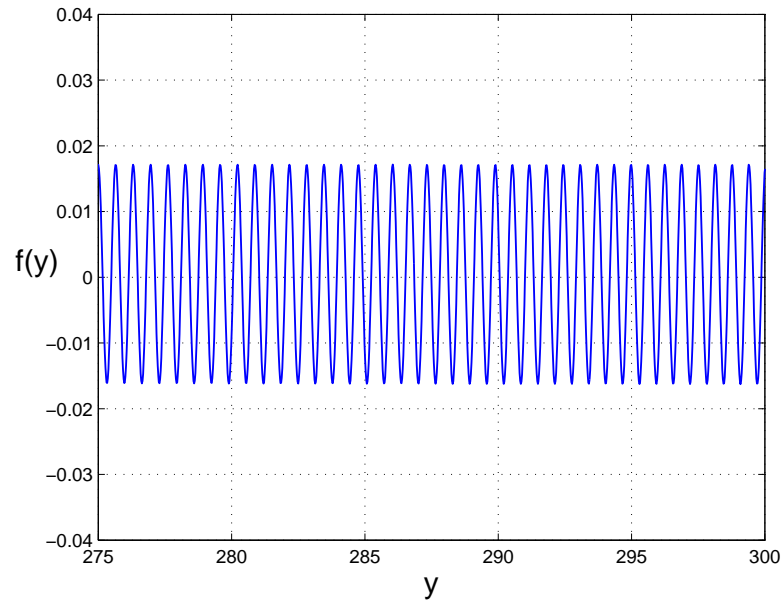


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-11: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.7$ .

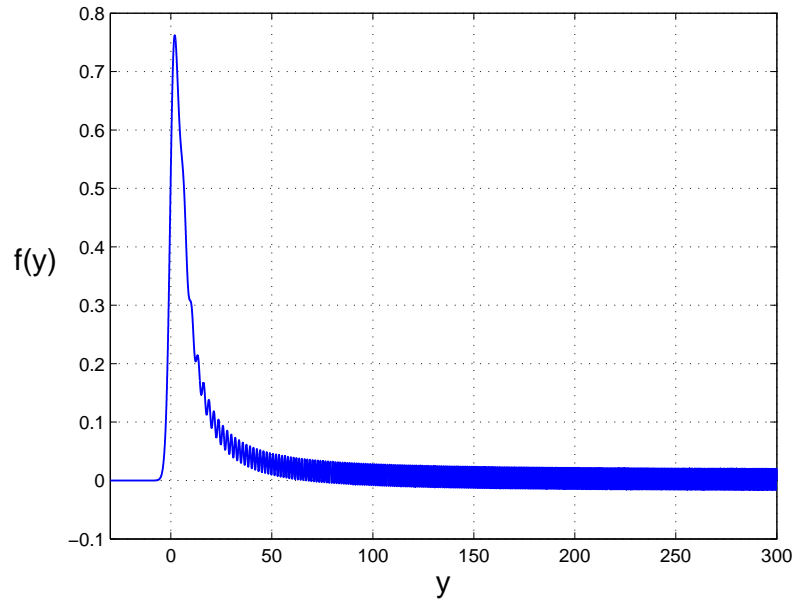


(a) Global solution structure.

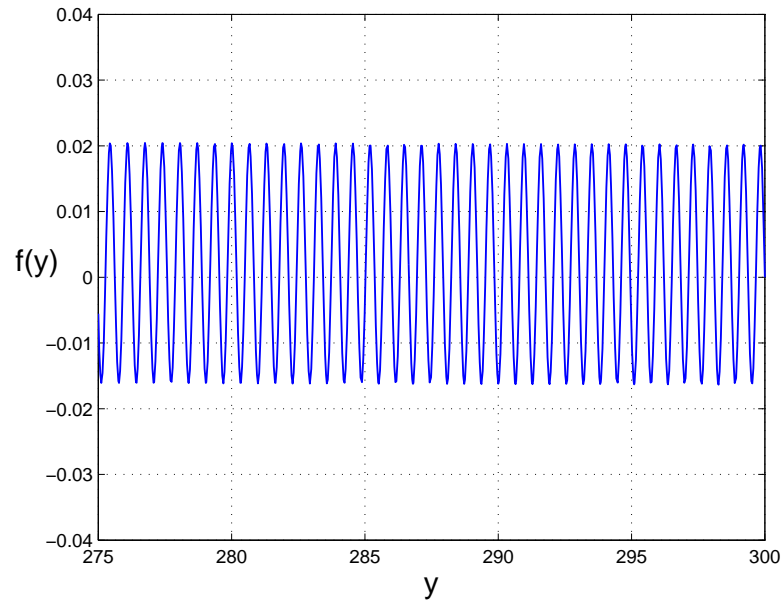


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-12: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 2.9$ .

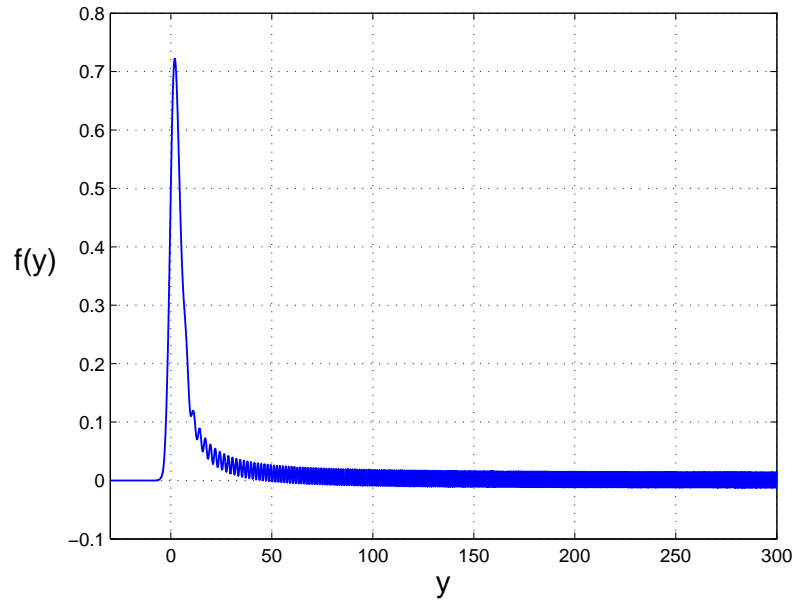


(a) Global solution structure.

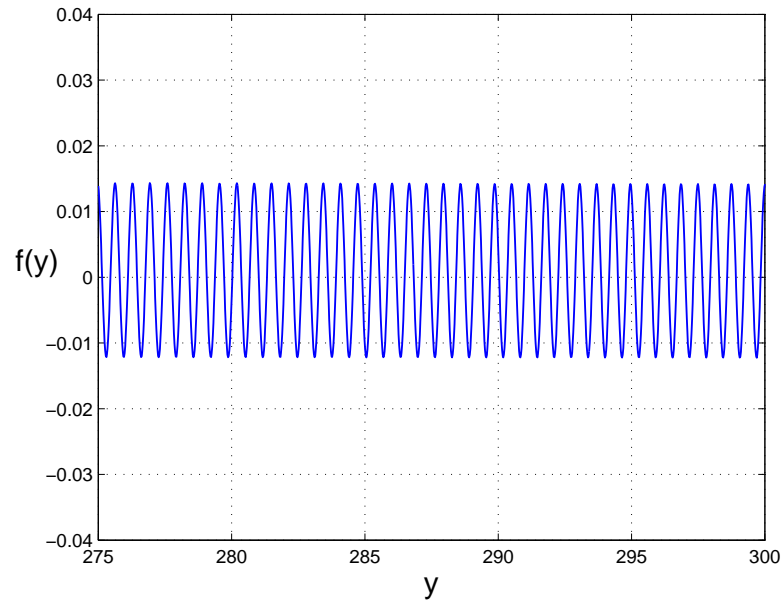


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-13: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 3$ .

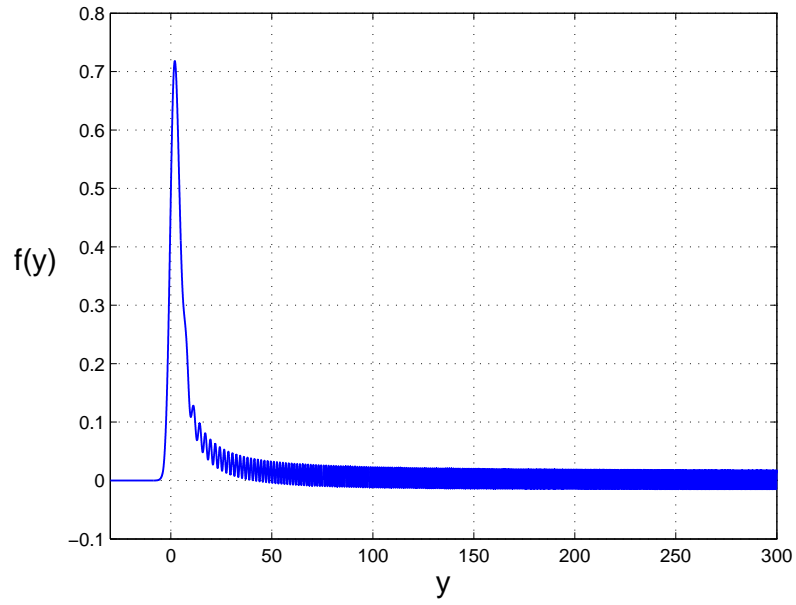


(a) Global solution structure.

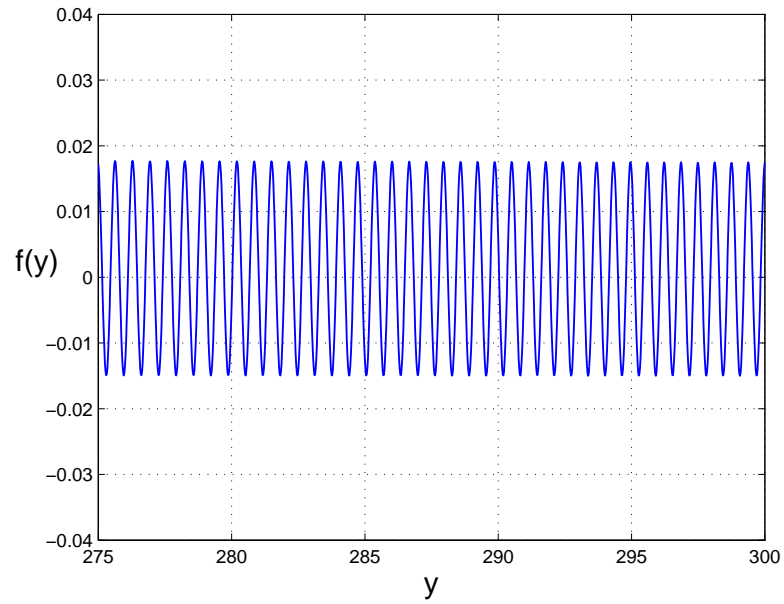


(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-14: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 3.1$ .



(a) Global solution structure.



(b) "Tail" of the solution, for  $y \gg 1$ .

Figure A-15: A very singular solution to  $u_t = u_{xxx} - u^p$ , with  $p = 3.2$ .

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